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RESPONSE OF AN ELASTIC DISK TO IMPACT AND MOVING LOADS†

By A. CEMAL ERINGEN‡ (*Purdue University, Division of Engineering Sciences, Lafayette, Indiana*)

[Received 20 December 1954]

SUMMARY

With the use of Fourier transforms a class of elasto-dynamic problems concerning disks have been solved. The disk is subjected to various types of dynamic loadings at the rim. The cases of impact and moving loads are studied in detail.

1. Introduction

Cylindrical roller bearings in high speed mechanisms are subject to dynamic loadings. Yet the usual design procedure is based on Hertz formulae which are the result of elasto-static considerations. In many other instances, gears, rollers, or disks are subject to impact or moving loads. If we neglect the coriolis terms we can also bring the rolling disks in contact into the category of disks subject to moving loads. Thus the aim of the present paper is to obtain the solution to this class of elasto-dynamic problems concerning the disk. The dynamic load is applied to the rim of the disk. Two normal concentrated dynamic loads at the two ends of a diameter moving or otherwise are special cases.

Some solutions of free oscillation of cylinders have been known since the time of Pochhammer (**1**), and later Pickett (**2**), J. Mindlin (**3**), T. Ghosh (**4**). Similarly, the problem of rotating disks has attracted attention of many authors (see, for instance, Lamb and Southwell (**5**), Timoshenko and Goodier (**6**), Love (**7**)). It seems, however, that the forced oscillation problems concerning disks and cylinders have escaped attention, excepting a paper by J. Mindlin (**8**), which consists of generalities in a problem related to ours.

The present method is applicable to ring problems and to the plane elasto-dynamic problems concerning circular holes. These problems will be treated in later papers.

† The research contained in this paper was obtained during investigations for a research project sponsored by the Office of Naval Research.

‡ Associate Professor of Division of Engineering Sciences.

2. Formulation of the problem

The equations of motion of plane homogeneous isotropic media in terms of plane polar coordinates r and θ and the time t are (**7**) (see Fig. 1)

$$\left.\begin{aligned} \gamma u_{,tt} &= (\lambda+2\mu)\Delta_{,r} - 2\mu r^{-1}\omega_{,\theta} \\ \gamma v_{,tt} &= (\lambda+2\mu)r^{-1}\Delta_{,\theta} + 2\mu\omega_{,r} \end{aligned}\right\}, \tag{1}$$

where $u(r,\theta,t)$ and $v(r,\theta,t)$ are components of the displacement vector, λ and μ are the Lamé constants, and γ is the mass density per unit volume.

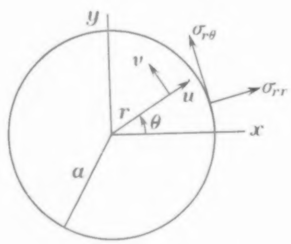

FIG. 1. Circular disk.

Dilatation Δ and rotation ω are related to u, v by

$$r\Delta = (ru)_{,r} + v_{,\theta}, \qquad 2r\omega = (rv)_{,r} - u_{,\theta}. \tag{2}$$

Subscripts after a comma represent differentiation, i.e. $u_{,\theta} = \partial u/\partial\theta$, etc.

Elimination of u and v between (1) and (2) leads to

$$\left.\begin{aligned} \alpha_1^2 r^2 \Delta_{,tt} &= r(r\Delta_{,r})_{,r} + \Delta_{,\theta\theta}, & \alpha_1^2 &= \gamma/(\lambda+2\mu) \\ \alpha_2^2 r^2 \omega_{,tt} &= r(r\omega_{,r})_{,r} + \omega_{,\theta\theta}, & \alpha_2^2 &= \gamma/\mu \end{aligned}\right\}. \tag{3}$$

These are the equations of dilatational and rotational waves.

Components σ_{rr}, $\sigma_{r\theta}$, $\sigma_{\theta\theta}$ of the stress tensor are given by:

$$\left.\begin{aligned} \sigma_{rr} &= \lambda\Delta + 2\mu u_{,r}, \qquad \sigma_{r\theta} = \mu r^{-1} u_{,\theta} + \mu r(v/r)_{,r} \\ \sigma_{\theta\theta} &= \lambda\Delta + 2\mu r^{-1}(v_{,\theta} + u) \end{aligned}\right\}. \tag{4}$$

The problem is to solve (1), (2) under a given σ_{rr} and $\sigma_{r\theta}$ at the rim $r = a$ of the disk:

$$\sigma_{rr}(a,\theta,t) = \sigma_0(\theta,t), \qquad \sigma_{r\theta}(a,\theta,t) = \tau_0(\theta,t) \tag{5}$$

subject to the condition that these surface tractions are in equilibrium at each instant.

3. The solution

The periodic solution of (3) with respect to θ is

$$\left.\begin{aligned} \bar{\Delta} &= \sum_{n=0}^{\infty} (A_{1n} \sin n\theta + A_{2n} \cos n\theta) Z_n(\rho_1) \\ \bar{\omega} &= \sum_{n=0}^{\infty} (B_{1n} \cos n\theta - B_{2n} \sin n\theta) Z_n(\rho_2) \end{aligned}\right\}, \tag{6}$$

$$Z_n(\rho_j) = C_{jn} J_n(\rho_j) + D_{jn} Y_n(\rho_j), \qquad \rho_j = \alpha_j \tau r \quad (j = 1, 2),$$

where $Z_n(\rho)$ is the cylinder function (**9**); A_{jn}, B_{jn}, C_{jn}, and D_{jn} are constants of integration; and the barred quantities represent the Fourier transforms, i.e.

$$\bar{F}(\tau) = \int_{-\infty}^{\infty} e^{i\tau t} F(t)\, dt. \tag{7}$$

The inversion formula for (7) is

$$F(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-it\tau} \bar{F}(\tau)\, d\tau. \tag{8}$$

Substituting (6) into (2) after taking the Fourier transforms of (2) and solving the resulting partial differential equations we obtain

$$\left.\begin{aligned} \bar{u} &= \sum_{n=0}^{\infty} U_{1n}(r) \sin n\theta + U_{2n}(r) \cos n\theta \\ \bar{v} &= \sum_{n=0}^{\infty} V_{1n}(r) \cos n\theta - V_{2n}(r) \sin n\theta \end{aligned}\right\}, \tag{9}$$

where

$$\left.\begin{aligned} -r^{-1} U_{jn}(r) &= A_{jn}\, \rho_1^{-1} Z_n'(\rho_1) + B_{jn}\, 2n \rho_2^{-2} Z_n(\rho_2) \\ -r^{-1} V_{jn}(r) &= A_{jn}\, n \rho_1^{-2} Z_n(\rho_1) + B_{jn}\, 2 \rho_2^{-1} Z_n'(\rho_2) \end{aligned}\right\}, \tag{10}$$

where primes represent differentiation.

Combining (4), (6), and (9) we obtain

$$\left.\begin{aligned} \bar{\sigma}_{rr}/2\mu = \sum_{n=0}^{\infty} [A_{1n} N_{1n}(r\tau) + B_{1n} N_{2n}(r\tau)] \sin n\theta + \\ + [A_{2n} N_{1n}(r\tau) + B_{2n} N_{2n}(r\tau)] \cos n\theta \\ \bar{\sigma}_{r\theta}/2\mu = \sum_{n=0}^{\infty} [A_{1n} S_{1n}(r\tau) + B_{1n} S_{2n}(r\tau)] \cos n\theta - \\ - [A_{2n} S_{1n}(r\tau) + B_{2n} S_{2n}(r\tau)] \sin n\theta \\ \bar{\sigma}_{\theta\theta}/2\mu = \sum_{n=0}^{\infty} [A_{1n} T_{1n}(r\tau) + B_{1n} T_{2n}(r\tau)] \sin n\theta + \\ + [A_{2n} T_{1n}(r\tau) + B_{2n} T_{2n}(r\tau)] \cos n\theta \end{aligned}\right\}. \tag{11}$$

where

$$\left.\begin{aligned}
N_{1n}(r\tau) &= (\lambda/2\mu)Z_n(\rho_1)+(1-n^2\rho_1^{-2})Z_n(\rho_1)+\rho_1^{-1}Z_n'(\rho_1)\\
N_{2n}(r\tau) &= 2n\rho_2^{-2}Z_n(\rho_2)-2n\rho_2^{-1}Z_n'(\rho_2)\\
S_{1n}(r\tau) &= n\rho_1^{-2}Z_n(\rho_1)-n\rho_1^{-1}Z_n'(\rho_1)\\
S_{2n}(r\tau) &= (1-2n^2\rho_2^{-2})Z_n(\rho_2)+2\rho_2^{-1}Z_n'(\rho_2)\\
T_{1n}(r\tau) &= (\lambda/2\mu)Z_n(\rho_1)+n^2\rho_1^{-2}Z_n(\rho_1)-\rho_1^{-1}Z_n'(\rho_1)\\
T_{2n}(r\tau) &= -N_{2n}(r\tau)
\end{aligned}\right\}. \tag{12}$$

4. The dynamic tractions applied to the rim of a disk

In the case of a disk, the stress and deformation components must be finite at $r = 0$. Hence $D_{jn} = 0$. Without loss of generality we also take $C_{jn} = 1$. This means that in all our formulae we must replace Z_n by J_n.

We use boundary conditions (5) to determine the constants A_{jn} and B_{jn}. Fourier's theorem thus leads to

$$\left.\begin{aligned}
A_{1n} &= [2\mu D_n(a\tau)]^{-1}[S_{2n}(a\tau)\tilde{\sigma}_{0s}-N_{2n}(a\tau)\tilde{\tau}_{0c}]\\
B_{1n} &= [2\mu D_n(a\tau)]^{-1}[-S_{1n}(a\tau)\tilde{\sigma}_{0s}+N_{1n}(a\tau)\tilde{\tau}_{0c}]\\
A_{2n} &= [2\mu D_n(a\tau)]^{-1}[S_{2n}(a\tau)\tilde{\sigma}_{0c}+N_{2n}(a\tau)\tilde{\tau}_{0s}]\\
B_{2n} &= [2\mu D_n(a\tau)]^{-1}[-S_{1n}(a\tau)\tilde{\sigma}_{0c}-N_{1n}(a\tau)\tilde{\tau}_{0s}] \quad (n = 1, 2,...)\\
A_{j0} &= \tfrac{1}{2}[A_{jn}]_{n=0}, \qquad B_{j0} = \tfrac{1}{2}[B_{jn}]_{n=0}\\
D_n(a\tau) &= N_{1n}(a\tau)S_{2n}(a\tau)-N_{2n}(a\tau)S_{1n}(a\tau)
\end{aligned}\right\}. \tag{13}$$

where

$$\tilde{\sigma}_{0c} = \pi^{-1}\int_0^{2\pi}\tilde{\sigma}_0(\theta,\tau)\cos n\theta\, d\theta,$$

$$\tilde{\sigma}_{0s} = \pi^{-1}\int_0^{2\pi}\tilde{\sigma}_0(\theta,\tau)\sin n\theta\, d\theta,$$

and $\tilde{\tau}_{0c}$ and $\tilde{\tau}_{0s}$ are similarly defined.

Various special cases are of interest:

(*a*) *Zero surface shear*: $\tilde{\tau}_{0c} = \tilde{\tau}_{0s} = 0$. (14)

(*b*) *Normal traction with central symmetry and* (*a*): $\tilde{\sigma}_{0s} = 0$. (15)

(*c*) (*b*) *with constant amplitude over* $0 \leqslant \theta \leqslant \alpha$, *i.e.*

$$(15) \quad \textit{and} \quad \tilde{\sigma}_0(\theta,\tau) = \bar{\sigma}(\tau)\begin{cases} 0 \leqslant \theta \leqslant \alpha \\ \pi-\alpha \leqslant \theta \leqslant \pi \end{cases}$$

$$= 0 \quad \alpha < \theta \leqslant \pi-\alpha. \tag{16}$$

We get $$\tilde{\sigma}_{0c} = 2\bar{\sigma}_0(\tau)(\pi n)^{-1}[1+(-)^n]\sin n\alpha. \tag{17}$$

(*d*) *Impact load and* (*c*):

With (16) and $\lim\limits_{\substack{\alpha\to 0\\ \bar{\sigma}_0\to\infty}} 2\bar{\sigma}_0(\tau)a\alpha = \bar{P}_0(\tau)$

we get

$$\sigma_{0c} = 2\bar{P}_0(\tau)/\pi a. \tag{18}$$

(*e*) *Impulsive concentrated load and* (*d*):

With (18) and $P_0(t) = P_0\,\delta(t)$, where $\delta(t)$ is the Dirac delta function defined by

$$\delta(t) = \begin{cases}\infty & t = 0\\ 0 & t \neq 0,\end{cases} \qquad \int\limits_{-\infty}^{\infty} \delta(t)\,dt = 1,$$

we find

$$\bar{P}_0(\tau) = P_0, \qquad \tilde{\sigma}_{0c} = 2P_0/\pi a, \tag{19}$$

where P_0 is the amplitude of the concentrated load.

5. Moving loads

(*a*) *Moving normal and tangential stresses*:

Moving loads can be represented by

$$\sigma_0(\theta,t) = \sigma_0(\theta-\Omega t), \qquad \tau_0(\theta,t) = \tau_0(\theta-\Omega_1 t). \tag{20}$$

Fourier transforms of these are

$$\bar{\sigma}_0(\theta,\tau) = 2e^{i\tau\theta/\Omega}s_0(\tau/\Omega), \qquad \tau_0(\theta,\tau) = 2e^{i\tau\theta/\Omega_1}s_1(\tau/\Omega_1), \tag{21}$$

$$s_0(\tau/\Omega) = \frac{1}{2\Omega}\int\limits_{-\infty}^{\infty} \sigma_0(\phi)e^{-i\tau\phi/\Omega}\,d\phi \quad \text{(in general)}$$

$$= \frac{1}{\Omega}\int\limits_{0}^{\infty} \sigma_0(\phi)\cos(\tau\phi/\Omega)\,d\phi \quad \text{when } \sigma_0(-\phi) = \sigma_0(\phi), \tag{22}$$

where $s_1(\tau/\Omega_1)$ follows from $s_0(\tau/\Omega)$ by replacing Ω and $\sigma_0(\phi)$ by Ω_1 and $\tau_0(\phi)$ respectively. Hence

$$\left.\begin{aligned}\tilde{\sigma}_{0c} &= (2\tau/\pi\Omega^2 i)[(\tau/\Omega)^2-n^2]^{-1}(e^{2\pi\tau i/\Omega}-1)s_0(\tau/\Omega)\\ \tilde{\sigma}_{0s} &= (2n/\pi\Omega)[(\tau/\Omega)^2-n^2]^{-1}(e^{2\pi\tau i/\Omega}-1)s_0(\tau/\Omega)\end{aligned}\right\}. \tag{23}$$

The quantities $\tilde{\tau}_{0c}$ and $\tilde{\tau}_{0s}$ are obtained from (23) by writing Ω_1 and $s_1(\tau/\Omega_1)$ in place of Ω and $s_0(\tau/\Omega)$ respectively.

(*b*) *Moving periodic loads*:

$$\left.\begin{aligned}\sigma_0(\theta-\Omega t) &= (Q_0/\pi a)\sum_{n=0}^{\infty} p_n\cos n(\theta-\Omega t)+q_n\sin n(\theta-\Omega t)\\ \tau_0(\theta-\Omega_1 t) &= (Q_0/\pi a)\sum_{n=0}^{\infty} r_n\cos n(\theta-\Omega_1 t)+s_n\sin n(\theta-\Omega_1 t)\end{aligned}\right\}. \tag{24}$$

After taking Fourier transforms of σ_0 and τ_0, we substitute in the two equations following (13) to obtain $\tilde{\sigma}_{0c},\ldots, \tilde{\tau}_{0s}$. This gives

$$\left.\begin{aligned} a\tilde{\sigma}_{0c}/Q_0 &= (p_n - iq_n)\delta(\Omega n-\tau)+(p_n+iq_n)\delta(-\Omega n-\tau) \\ a\tilde{\sigma}_{0s}/Q_0 &= (ip_n+q_n)\delta(\Omega n-\tau)+(-ip_n+q_n)\delta(-\Omega n-\tau) \end{aligned}\right\}. \tag{25}$$

The quantities $\tilde{\tau}_{0c}$ and $\tilde{\tau}_{0s}$ follow from (25) by writing r_n and s_n in place of p_n and q_n. In obtaining (25) we have used the formal relation† (**10**)

$$2\pi\delta(u) = \int\limits_{-\infty}^{\infty} e^{-itu}\,dt.$$

Components of displacement and stress tensor can now be obtained by combining (9), (11), (13), and (25) and taking inverse Fourier transforms. Thus

$$\left.\begin{aligned} -(2\pi a\mu/Q_0 r)u &= \sum_{n=0}^{\infty} u_n^{(1)}(r\Omega n)[p_n\cos n(\theta-\Omega t)+q_n\sin n(\theta-\Omega t)]+ \\ &\quad + u_n^{(2)}(r\Omega n)[-r_n\sin n(\theta-\Omega_1 t)+s_n\cos n(\theta-\Omega_1 t)] \\ -(2\pi a\mu/Q_0 r)v &= \sum_{n=0}^{\infty} v_n^{(1)}(r\Omega n)[-p_n\sin n(\theta-\Omega t)+q_n\cos n(\theta-\Omega t)]+ \\ &\quad + v_n^{(2)}(r\Omega n)[-r_n\cos n(\theta-\Omega_1 t)-s_n\sin n(\theta-\Omega_1 t)] \\ (\pi a/Q_0)\sigma_{rr} &= \sum_{n=0}^{\infty} \sigma_{1n}^{(1)}(r\Omega n)[p_n\cos n(\theta-\Omega t)+q_n\sin n(\theta-\Omega t)]+ \\ &\quad + \sigma_{1n}^{(2)}(r\Omega_1 n)[-r_n\sin n(\theta-\Omega_1 t)+s_n\cos n(\theta-\Omega_1 t)] \\ (\pi a/Q_0)\sigma_{r\theta} &= \sum_{n=0}^{\infty} \tau_n^{(1)}(r\Omega n)[-p_n\sin n(\theta-\Omega t)+q_n\cos n(\theta-\Omega t)]+ \\ &\quad + \tau_n^{(2)}(r\Omega n)[-r_n\cos n(\theta-\Omega_1 t)-s_n\sin n(\theta-\Omega_1 t)] \\ (\pi a/Q_0)\sigma_{\theta\theta} &= \sum_{n=0}^{\infty} \sigma_{2n}^{(1)}(r\Omega n)[p_n\cos n(\theta-\Omega t)+q_n\sin n(\theta-\Omega t)]+ \\ &\quad + \sigma_{2n}^{(2)}(r\Omega n)[-r_n\sin n(\theta-\Omega_1 t)+s_n\cos n(\theta-\Omega_1 t)] \end{aligned}\right\} \tag{26}$$

where

$$\left.\begin{aligned} u_n^{(1)}(r\Omega n) &= [D_n(a\Omega n)]^{-1}[(\alpha_1 r\Omega n)^{-1}J_n'(\alpha_1 r\Omega n)S_{2n}(a\Omega n)- \\ &\qquad -2n(\alpha_2 r\Omega n)^{-2}J_n(\alpha_2 r\Omega n)S_{1n}(a\Omega n)] \\ v_n^{(1)}(r\Omega n) &= [D_n(a\Omega n)]^{-1}[n(\alpha_1 r\Omega n)^{-2}J_n(\alpha_1 r\Omega n)S_{2n}(a\Omega n)- \\ &\qquad -2(\alpha_2 r\Omega n)^{-1}J_n'(\alpha_2 r\Omega n)S_{1n}(a\Omega n)] \\ \sigma_{1n}^{(1)}(r\Omega n) &= [D_n(a\Omega n)]^{-1}[N_{1n}(r\Omega n)S_{2n}(a\Omega n)-N_{2n}(r\Omega n)S_{1n}(a\Omega n)] \\ \tau_n^{(1)}(r\Omega n) &= [D_n(a\Omega n)]^{-1}[S_{1n}(r\Omega n)S_{2n}(a\Omega n)-S_{2n}(r\Omega n)S_{1n}(a\Omega n)] \\ \sigma_{2n}^{(1)}(r\Omega n) &= [D_n(a\Omega n)]^{-1}[T_{1n}(r\Omega n)S_{2n}(a\Omega n)-T_{2n}(r\Omega n)S_{1n}(a\Omega n)] \end{aligned}\right\} \tag{27}$$

† This is justified in the sense of a distribution function (**11**).

and where

$$D_n(a\Omega n) = N_{1n}(a\Omega n)S_{2n}(a\Omega n) - N_{2n}(a\Omega n)S_{1n}(a\Omega n). \tag{28}$$

The functions $u_n^{(2)}$, $v_n^{(2)}$, $\sigma_{1n}^{(2)}$, $\tau_n^{(2)}$ and $\sigma_{2n}^{(2)}$ are obtained from the corresponding ones with superscript (1) by replacing Ω by Ω_1 and $S_{2n}(a\Omega n)$ and $S_{1n}(a\Omega n)$ by $N_{2n}(a\Omega_1 n)$ and $N_{1n}(a\Omega_1 n)$ respectively, except in $D_n(a\Omega n)$, where we replace Ω by Ω_1.

(*c*) *Two diametrically opposite, moving, concentrated loads* (Fig. 2). In this case, which is one of technical importance, we have $\tau_0 = 0$, hence $r_n = s_n = 0$. The concentrated loads can formally be represented by

$$\begin{aligned}\sigma_0(\theta-\Omega t) &= (Q_0/\pi a)[\delta(\theta-\Omega t)+\delta(\pi-\theta+\Omega t)] \\ &= (Q_0/\pi a)\sum_{n=0}^{\infty} p_n \cos n(\theta-\Omega t)+q_n \sin n(\theta-\Omega t),\end{aligned} \tag{29}$$

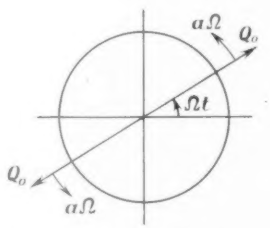

FIG. 2. Moving radial load.

where Q_0 is the amplitude of each of the concentrated radial loads. From (29) the Fourier coefficients p_n and q_n are

$$p_n = \begin{cases} 2/\pi & \text{for } n \text{ even,} \\ 0 & \text{for } n \text{ odd.} \end{cases} \qquad q_n = 0. \tag{30}$$

Hence (26), together with (29) and (30), and $r_n = s_n = 0$, gives the displacement and stress components. Below we give displacement components. The rest is obtained in an obvious manner.

$$\left.\begin{aligned} -(\pi^2 a\mu/Q_0 r)u &= \sum_{0,2,4,\ldots} u_n^{(1)}(r\Omega n)\cos n(\theta-\Omega t) \\ (\pi^2 a\mu/Q_0 r)v &= \sum_{0,2,4,\ldots} v_n^{(1)}(r\Omega n)\sin n(\theta-\Omega t) \end{aligned}\right\}, \tag{31}$$

where $u_n^{(1)}$ and $v_n^{(1)}$ are given by (27).

Concentrated moving shear loads and other types of load combination may easily be obtained from (24) and (26).

6. Computation and discussion

Computations have been carried out to determine the roots of

$$D_n(a\Omega n) = 0$$

given by (28). The roots of this equation give the resonance speed for the concentrated load.

The ratio of the resonance speed $C_r = a\Omega$ of the concentrated load to the dilatational wave velocity $C_1 = \alpha_1^{-1}$ has been solved from $D_n = 0$ for a steel cylinder with $E = 30 \times 10^6$ p.s.i. and $\nu = 0{\cdot}3$. The first four roots of $D_n = 0$ for $n = 1, 2, 3, 4$ are listed in the table given below.†

The computation was carried out on I.B.M. Card Program Calculator and are correct up to two decimal places.

The values of $nC_r/C_1 = n\alpha_1\Omega$ (obtained by calculating the roots of $D_n(a\Omega n) = 0$). Steel: $E = 30 \times 10^6$, $\nu = 0{\cdot}3$

n	$D_1 = 0$	$D_2 = 0$	$D_3 = 0$	$D_4 = 0$
1	1·51	1·26	1·94	2·52
2	3·43	2·35	3·21	4·08
3	3·80	4·22	4·95	5·67
4	5·33	5·06	6·22	7·19

From this table it is seen that the smallest C_r/C_1 is obtained for $n = 2$. This value is $C_r/C_1 = 1{\cdot}26/2 = 0{\cdot}63$. We therefore expect a critical speed for the moving load in the neighbourhood of $0{\cdot}63C_1$ which will create resonance in the disk.

The Rayleigh surface wave velocity for $\nu = 0{\cdot}29$ is $0{\cdot}9258$ times the velocity of shear waves or $0{\cdot}503$ times the velocity of dilational waves. Thus $0{\cdot}63C_1$ represents a velocity between the Rayleigh surface wave velocity and the shear wave velocity. Further we notice that this velocity is minimum for $n = 2$ rather than $n = 1$. Examining (31) we find that we have only even terms. Hence diametrically opposite moving loads give a smaller critical speed than a single load. In the former case the fundamental mode $n = 0$ represents a uniform lateral extension which is not dependent on time. Thus it is static in nature and has no resonance frequency associated with it. The second mode is of $\cos 2\theta$ type and gives the minimum critical speed mentioned above. It is interesting to note that the minimum critical speed in the case of a flat semi-infinite plate (the Rayleigh surface wave velocity) is less than that of the curved surface (the disk in the present case). With this point in mind perhaps one can classify the curved surfaces from the elasto-dynamic point of view, though this process will, no doubt, present mathematical difficulties.

† For $n = 0$ we take the limit of the functions in (27) and find finite values.

The present results, of course, must be accepted tentatively until verified experimentally.

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THE EFFECT OF VISCOSITY UPON THE CRITICAL FLOW OF A LIQUID THROUGH A CONSTRICTION

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SUMMARY

The motions of a viscous liquid under gravity over a broad-crested weir, and with swirl under pressure through a convergent-divergent nozzle, are analysed on the supposition that the whole of the discharge passes through a laminar boundary layer. The constrictions are assumed to be gradual; and the pressure gradients in the direction of streaming being favourable, the velocity distributions over the cross-sections are taken to be of polynomial forms. When the momentum integral equations are satisfied, first-order differential equations are obtained which show that, as with inviscid flow, the motion is 'critical' in the sense that arbitrary inlet conditions may not be specified. The flow must be determined from the conditions that prevail at the critical cross-sections, which are displaced downstream from the throats of the constrictions. The equations are difficult to solve because at the critical cross-sections, where their step-by-step integration has to be begun, they take the indeterminate form 0/0, but a numerical example of each type of motion is given. It is proved that a long wave of small amplitude can maintain itself stationary on the moving liquid at the critical cross-sections. A comparison is made with inviscid flow through both kinds of constriction.

1. Introduction

Among the well-known examples of the 'critical' flow of a liquid are flow under gravity over a broad-crested weir and swirling flow under pressure through a convergent-divergent nozzle. These motions have been examined by several investigators, and a theoretical review was given by Binnie (1) who showed that, in general, the flow sets itself in such a way that the streaming velocity is equal at the throat to the velocity of a long wave. This analysis is confined to an inviscid liquid; and, as in Reynolds's classical treatment of gas flow through a nozzle, the constriction is supposed to be gradual so that transverse components of velocity and acceleration may be neglected.

The present paper is an inquiry into the consequences of assuming that the liquid is viscous instead of perfect and that the whole of the flow passes through a laminar boundary layer, the inlet length being ignored. The working is approximate in that, following Pohlhausen, we take the velocity distributions in the boundary layers to be of reasonable polynomial forms, and employ von Kármán's method of satisfying the momentum integrals.

The two cases mentioned above are examined in detail. Although it is of no practical importance, the broad-crested weir is worth considering

because the analysis is relatively simple; thus the basic ideas may readily be understood and numerical results obtained with comparative ease. For swirling flow through a nozzle, Taylor (**2**) made it clear that in many practical cases the boundary layer is so thick that most of the discharge passes through it. This conclusion was supported by Binnie and Harris (**3**), who analysed the motion for an inviscid liquid and found that, if the swirl were sufficiently great, the discharge was reduced to zero, all the energy in the supply being used up in producing tangential velocity. It is a fact of common observation that the discharge does not cease in this way, hence under these conditions boundary layer flow becomes predominant, as in some of the experiments on gravity flow through a trumpet described by Binnie and Hookings (**4**). The method of solving the weir problem is therefore applied to the nozzle, but the working is much more complicated, for two momentum integrals must be satisfied, yielding a pair of simultaneous differential equations. In both cases an outline of the theory for frictionless motion is included, partly for purposes of comparison and partly to demonstrate that at the throats the derivatives of the velocities and the thicknesses of the streams take the indeterminate form 0/0.

2. Frictionless flow over a broad-crested weir

We consider unit width of the broad-crested weir outlined in side elevation in Fig. 1. The liquid passes steadily from a large reservoir, the surface of

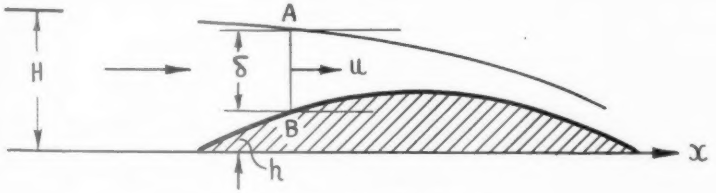

FIG. 1. Inviscid flow over weir.

which stands at a level H above a horizontal datum, over a weir (shown shaded), and falls away freely on the downstream side. At a typical cross-section, the depth AB of the stream is denoted by δ and the height of the weir above the datum by h. Horizontal distances are given by the coordinate x. The vertical component of velocity being ignored, the horizontal velocity u is easily shown to be uniform over AB, and its magnitude is obtained from Bernoulli's equation

$$H = \frac{u^2}{2g} + \delta + h, \tag{2.1}$$

in which g is the acceleration due to gravity. The equation of continuity is

$$Q = u\delta, \tag{2.2}$$

where Q is the discharge per unit width. On differentiating (2.1) and (2.2) with respect to x, we obtain

$$0 = \frac{u}{g}\frac{du}{dx}+\frac{d\delta}{dx}+\frac{dh}{dx}, \tag{2.3}$$

and

$$0 = u\frac{d\delta}{dx}+\delta\frac{du}{dx}. \tag{2.4}$$

When du/dx is eliminated from (2.3) and (2.4), the result is

$$\frac{d\delta}{dx} = -\frac{\dfrac{dh}{dx}}{1-\dfrac{u^2}{g\delta}} \tag{2.5}$$

$$= -\frac{\delta^3\dfrac{dh}{dx}}{\delta^3-\dfrac{Q^2}{g}}. \tag{2.6}$$

We are concerned here with the type of motion in which the liquid is accelerated throughout its passage over the weir, therefore $d\delta/dx$ is negative everywhere. In the approach to the throat dh/dx is positive, and in that region $u^2/(g\delta)$ must be less than unity. At the throat defined by $dh/dx = 0$, $d\delta/dx \neq 0$, therefore from (2.5)

$$u_t^2 = g\delta_t, \tag{2.7}$$

in which the suffix t denotes throat values. It follows from (2.1) and (2.7) that

$$H-h_t = \tfrac{3}{2}\delta_t. \tag{2.8}$$

Then, if δ_t and the variation of h with x are specified, the profile of the stream can be determined from the relation

$$h_t+\tfrac{3}{2}\delta_t-h = \frac{\delta_t^3}{2\delta^2}+\delta, \tag{2.9}$$

which is obtained from (2.1), (2.2), (2.7), and (2.8).

Another, but more cumbersome, way of evaluating δ is also possible. If $\delta_t = (Q^2/g)^{\frac{1}{3}}$ is specified, we could integrate (2.6) commencing at the throat, but there $d\delta/dx$ takes the form 0/0. On applying to (2.6) the usual method of dealing with this condition, we find that for this section

$$\frac{d\delta}{dx} = -\left(-\frac{\delta}{3}\frac{d^2h}{dx^2}\right)^{\frac{1}{2}}, \tag{2.10}$$

and the integration of (2.6) may be begun with this value.

If from (2.3) and (2.4) we remove $d\delta/dx$, we obtain

$$\frac{du}{dx} = \frac{u}{\delta}\frac{\dfrac{dh}{dx}}{1-\dfrac{u^2}{g\delta}}, \tag{2.11}$$

and this expression also takes the form $0/0$ at the throat.

3. Viscous flow over a broad-crested weir

The entire thickness of the stream is now taken to be influenced by viscosity. We employ the same notation as before, adding (as indicated

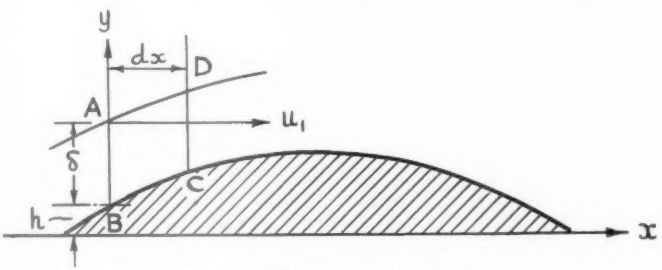

FIG. 2. Viscous flow over weir.

in Fig. 2) u_1 to denote the horizontal velocity at the free surface and y to denote vertical distance above the constriction. It will be assumed that the distribution of velocity over AB has the same form wherever AB may be situated; and, writing η for y/δ, we may suppose u to be given by

$$u = u_1(2\eta-2\eta^3+\eta^4), \tag{3.1}$$

for this relation satisfies the conditions

$$\left.\begin{aligned} u &= 0 \quad \text{at } \eta = 0\\ u &= u_1 \quad \text{at } \eta = 1\\ \frac{\partial u}{\partial y} &= 0 \quad \text{at } \eta = 1 \end{aligned}\right\}. \tag{3.2}$$

Then

$$Q = \int\limits_0^\delta u\,dy = \alpha u_1\delta, \quad \text{where } \alpha = 0{\cdot}7; \tag{3.3}$$

$$\int\limits_0^\delta u^2\,dy = \beta u_1^2\delta, \quad \text{where } \beta = 0{\cdot}5825; \tag{3.4}$$

$$\left[\frac{\partial u}{\partial y}\right]_{y=0} = \gamma\frac{u_1}{\delta}, \quad \text{where } \gamma = 2. \tag{3.5}$$

Alternatively, in place of (3.1), we may adopt the simpler form

$$u = u_1(2\eta-\eta^2), \tag{3.6}$$

which has the same shape as the full line in Fig. 4. This relation conforms to (3.2) and yields

$$\alpha = 0{\cdot}\dot{6}, \qquad \beta = 0{\cdot}5\dot{3}, \qquad \gamma = 2. \tag{3.7}$$

There is thus no great difference between the two sets of numerical coefficients.

The horizontal momentum integral can be obtained from first principles by considering the horizontal force exerted on the element $ABCD$ shown in Fig. 2. The density, viscosity, and kinematic viscosity of the liquid being ρ, μ, and ν, this force, reckoned positive in the direction of streaming, has three components:

(i) the difference between the hydrostatic forces on AB and DC, which is $-g\rho\delta\,(d\delta/dx)\,dx$;

(ii) the horizontal component of the normal pressure exerted by BC, which is $-g\rho\delta(dh/dx)\,dx$;

(iii) the viscous force on BC, which is $-\mu[\partial u/\partial y]_{y=0}\,dx$.

Therefore the momentum integral is

$$\frac{\partial}{\partial x}\int\limits_0^\delta u^2\,dy = -g\delta\frac{d\delta}{dx}-g\delta\frac{dh}{dx}-\nu\left[\frac{\partial u}{\partial y}\right]_{y=0}, \tag{3.8}$$

which with the aid of (3.4) and (3.5) becomes

$$\beta\frac{\partial}{\partial x}(u_1^2\delta) = -g\delta\frac{d\delta}{dx}-g\delta\frac{dh}{dx}-\nu\gamma\frac{u_1}{\delta}. \tag{3.9}$$

From (3.9) u_1 is removed by means of (3.3), and we obtain

$$\frac{d\delta}{dx} = -\frac{\delta^3(dh/dx)+T}{\delta^3-S}, \tag{3.10}$$

where

$$S = \frac{\beta Q^2}{\alpha^2 g}, \qquad T = \frac{\nu\gamma Q}{\alpha g}. \tag{3.11}$$

This equation for determining δ should be compared with (2.6), which is the corresponding result for inviscid flow.

Again $d\delta/dx$ is negative over the whole of the weir. Now since dh/dx is positive on the upstream side of the geometrical throat and negative on the downstream side, there is a certain cross-section where the numerator of (3.10) changes sign. At this section, which may be termed the effective throat, the denominator must change sign also; or, in other words, $d\delta/dx$ takes the form $0/0$. Enforcing this condition we have

$$\delta^3\frac{dh}{dx} = -T, \quad\text{and}\quad \delta^3 = S. \tag{3.12}$$

The second result can be transformed to

$$u_1^2 = \frac{g\delta}{\beta}, \tag{3.13}$$

which is similar to (2.7), and again a definite relation exists between the depth and the surface velocity at the throat. From (3.12) it appears that

$$\frac{dh}{dx} = -\frac{T}{S} = -\frac{\nu\alpha\gamma}{\beta Q} = -\frac{\nu\gamma}{(\beta g\delta^3)^{\frac{1}{2}}}. \tag{3.14}$$

Thus the position of the effective throat, which is determined by (3.14), is shifted from the geometrical throat in the opposite direction to that of the external force due to viscosity. This result agrees with the analysis, given by Binnie (**1**), of the swirling flow of an inviscid liquid passing under gravity down a vertical trumpet; in that case the external force lay in the direction of streaming, and the effective throat was found to be moved upwards, although the displacement was very small in the numerical examples considered by Binnie and Hookings (**4**).

The integration of (3.10) is begun at the effective throat, and again, before a start can be made, the difficulty due to the 0/0 form must be overcome. On applying the usual method to (3.10) we obtain

$$\left(\frac{d\delta}{dx}\right)^2 + \frac{dh}{dx}\frac{d\delta}{dx} + \frac{\delta}{3}\frac{d^2h}{dx^2} = 0. \tag{3.15}$$

Since dh/dx and d^2h/dx^2 are negative, this quadratic equation possesses only one negative root and no ambiguity arises.

If δ is eliminated instead of u_1 from (3.9), we arrive at an expression for du/dx, which can be changed to

$$\frac{du_1}{dx} = \frac{u_1}{\delta}\,\frac{\delta^3(dh/dx)+T}{\delta^3-S}. \tag{3.16}$$

This bears the same resemblance to (3.10) as (2.11) does to (2.5), and it takes the form 0/0 at the effective throat.

With the same assumptions and methods as before, it may readily be shown that (3.13) also gives the relation between the depth and surface velocity of a horizontal stream on which a long wave of infinitesimal amplitude can maintain itself stationary. We take δ (now constant) to be the depth of the undisturbed stream, and consider a vertical cross-section where u' is the small additional surface velocity due to a wave of elevation ϵ. Across this section the velocity distribution is assumed to be given by

$$u = (u_1+u')\left\{\frac{2y}{\delta+\epsilon} - 2\left(\frac{y}{\delta+\epsilon}\right)^3 + \left(\frac{y}{\delta+\epsilon}\right)^4\right\} \tag{3.17}$$

in accordance with (3.1), therefore from (3.3) the discharge is

$$Q = \int_0^{\delta+\epsilon} u\,dy = \alpha(u_1+u')(\delta+\epsilon). \tag{3.18}$$

Similarly, for the undisturbed stream

$$Q = \int_0^{\delta} u_1\left\{2\frac{y}{\delta}-2\left(\frac{y}{\delta}\right)^3+\left(\frac{y}{\delta}\right)^4\right\} dy = \alpha u_1\delta, \tag{3.19}$$

hence

$$u_1+u' = \frac{u_1\delta}{\delta+\epsilon}. \tag{3.20}$$

In applying (3.8) to this problem, we notice that dh/dx is now zero, and we neglect the last term, supposing that it is balanced by some external force which maintains the steady motion. Using (3.17), (3.4), and (3.20), we have

$$\int_0^{\delta+\epsilon} u^2\,dy = \beta(u_1+u')^2(\delta+\epsilon) = \beta u_1^2\delta\left(1-\frac{\epsilon}{\delta}\right), \tag{3.21}$$

thus (3.8) reduces to

$$-\beta u_1^2\frac{d\epsilon}{dx} = -g\delta\frac{d\epsilon}{dx}, \tag{3.22}$$

or

$$u_1^2 = \frac{g\delta}{\beta} \tag{3.23}$$

in agreement with (3.13).

4. Numerical example of flow over a broad-crested weir

The weir is taken to be of symmetrical parabolic shape defined by

$$h = 10^{-4}(200x-x^2). \tag{4.1}$$

Thus it extends from $x = 0$ to $x = 200$ with a maximum rise equal to unity. The critical depth is assumed to be 0·1 and to occur at $x = 135$; and there dh/dx, which is given by

$$\frac{dh}{dx} = 2\times10^{-4}(100-x), \tag{4.2}$$

has the value $-0{\cdot}007$. It follows from (3.12) that T must then be 7×10^{-6}. If the foregoing figures are in inches, this requirement is met by water at about 17° C. for which $\nu = 1{\cdot}660\times10^{-3}$ in.^{2}/sec. For from (3.12) and (3.14) T can be written in the form

$$T = \nu\gamma\left(\frac{\delta^3}{\beta g}\right)^{\frac{1}{2}}; \tag{4.3}$$

and on inserting into this the above values of δ and ν together with $\beta = 0{\cdot}5825$ and $\gamma = 2$ given in (3.4) and (3.5), we obtain the result aimed at. Accordingly, with the aid of (4.2), (3.10) becomes

$$\frac{d\delta}{dx} = -\frac{2\times10^{-4}(100-x)\delta^3+7\times10^{-6}}{\delta^3-0{\cdot}001}, \tag{4.4}$$

which from the initial values $\delta = 0{\cdot}1$ in., $x = 135$ in., is to be integrated in both directions as far as $x = 0$ and $x = 200$ in. The solution has very kindly been obtained for me by Dr. J. C. P. Miller, whose result is given in column 3 of Table I. The addition of the corresponding values of h yields the satisfactory profile of the stream shown in column 5. It will be seen that at small values of x, in the region most remote from the start of the calculation, the surface is very slowly falling; and, as it should, the slope continuously increases.

To afford a comparison, an example of frictionless flow has been added to Table 1 by means of (2.9). In this the value of δ_t at $x = 100$ has been so chosen that, at $x = 135$, δ is the same as in the viscous case. From the results set out in columns 4 and 6 it appears that, as expected, the profile is steeper, and the effect of viscosity in holding up the stream is well marked at the downstream end of the weir. For the first two entries in column 3 differ by 0·1901 and those in column 4 by 0·1907, whereas the last two entries differ by 0·00258 and 0·00444 respectively.

TABLE I

Depths and profiles (in inches) of viscous and inviscid streams passing over weir

		δ		$\delta + h$	
x	h	*Viscous*	*Inviscid*	*Viscous*	*Inviscid*
1	2	3	4	5	6
0	0·	1·2064	1·2686	1·2064	1·2686
10	·19	1·0163	1·0779	1·2063	1·2679
20	·36	0·8459	0·9068	1·2059	1·2668
30	·51	·6955	·7553	1·2055	1·2653
40	·64	·5646	·6229	1·2046	1·2629
50	·75	·45322	·50910	1·20322	1·25910
60	·84	·36074	·41325	1·20074	1·25325
70	·91	·28633	·33418	1·19633	1·24418
80	·96	·22858	·27032	1·18858	1·23032
90	0·99	·18541	·21975	1·17541	1·20975
100	1·00	·15413	·18027	1·15413	1·18027
110	0·99	·13187	·14970	1·12187	1·13970
120	·96	·11603	·12605	1·07603	1·08605
130	·91	·10457	·10765	1·01457	1·01765
135	·8775	·10000	·10000	0·97750	0·97750
140	·84	·09603	·09320	·93603	·93320
150	·75	·08948	·08171	·83948	·83171
160	·64	·08429	·07246	·72429	·71246
170	·51	·08007	·06490	·59007	·57490
180	·36	·07655	·05864	·43655	·41864
190	·19	·07357	·05341	·26357	·24341
200	0·	·07099	·04897	·07099	·04897

From the table some estimate of the errors involved in the approximation (3.1) may be obtained. The equation of motion corresponding to (3.8) is

$$u\frac{\partial u}{\partial x} = -g\frac{d}{dx}(\delta+h)+\nu\frac{\partial^2 u}{\partial y^2}. \tag{4.5}$$

At the solid boundary (3.1) makes $\partial^2 u/\partial y^2$ equal to zero, whereas the right-hand side of (4.5) should vanish. To meet this boundary condition as well as those given by (3.2) the profile

$$u = u_1(A\eta+B\eta^2+C\eta^3) \tag{4.6}$$

was tried at $x = 130$. At this section $u_1 = 7{\cdot}788$ in./sec. from (3.3) and (3.13), and with the aid of the third-difference formula given in *Interpolation and Allied Tables* (**6**) the derivative of $\delta+h$ was found to be $-0{\cdot}006980$. After A, B, and C had been evaluated, the calculations yielded $\alpha = 0{\cdot}6725$, $\beta = 0{\cdot}5404$, $\gamma = 2{\cdot}0705$. These results should be compared with (3.3)–(3.5). The effect of imposing the correct boundary condition is small at this section, but the final entries in column 5 show large changes in $\delta+h$, and there the discrepancy is greater. In that region, however, another source of error is least. Equation (3.1) supposes most of the kinetic energy to be concentrated near the free surface, but in inviscid flow the whole cross-section of the stream has equal acceleration, and with viscosity in operation the same tendency is to be expected. The redistribution by viscous action is not immediate as is assumed in (3.1), but the error becomes smaller as the state of equilibrium between the gravity and the viscous forces is more nearly attained. Column 3 shows that at the downstream end of the constriction the alterations in δ, and consequently in u, are proportionately less than elsewhere. The motion is tending towards steady flow down an inclined plane, for which the exact velocity distribution is known to be given by (3.6).

5. Frictionless swirling flow through a convergent-divergent nozzle

We consider steady flow from a reservoir maintained at a pressure head H, which is sufficiently high for gravity effects to be negligible, through the circular nozzle shown in Fig. 3. The tangential velocity v at any radius r is related to the swirl constant Ω by the equation

$$v = \frac{\Omega}{r}. \tag{5.1}$$

At a typical cross-section, the radii of the nozzle and of the core are denoted by a and b, and the radial thickness δ of the stream is given by

$$\delta = a-b. \tag{5.2}$$

It is convenient to write

$$v_1 = \frac{\Omega}{b} \tag{5.3}$$

for the tangential velocity at the core. To obtain non-dimensional results we take

$$R = \frac{\delta}{a}, \tag{5.4}$$

and

$$K = \frac{Q^2}{v_1^2 a^4}, \tag{5.5}$$

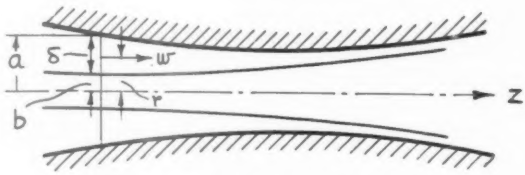

FIG. 3. Inviscid flow through nozzle.

where Q is the discharge through a sector subtending one radian at the axis. The radial velocity being neglected, the axial velocity w is uniform over the cross-section and is found from Bernoulli's equation, which takes the form

$$2gH = w^2+v_1^2. \tag{5.6}$$

Surface tension effects over the core are assumed to be imperceptible, so that the pressure there is zero. The equation of continuity is

$$Q = \frac{w}{2}(a^2-b^2). \tag{5.7}$$

On inserting (5.2), (5.4), and (5.5) into (5.6) and (5.7) and differentiating with respect to the axial coordinate z, we obtain two equations, which lead after some manipulation to

$$\frac{d\delta}{dz} = -\frac{\dfrac{da}{dz}\left(1+\dfrac{8K(1-R)}{R^2(2-R)^3}\right)}{\dfrac{8K(1-R)^2}{R^3(2-R)^3}-1} \tag{5.8}$$

and

$$\frac{a}{v_1}\frac{dv_1}{dz} = -\frac{\dfrac{da}{dz}\,\dfrac{8K}{R^3(2-R)^3}}{\dfrac{8K(1-R)^2}{R^3(2-R)^3}-1}. \tag{5.9}$$

It was shown by Binnie and Harris (**3**) that at the throat dw/dz takes the $0/0$ form and w is given by

$$w^2 = \frac{\Omega^2(a^2-b^2)}{2b^4}. \tag{5.10}$$

From (5.10) it can be deduced that the denominators of (5.8) and (5.9) are then zero, hence $d\delta/dz$ and dv_1/dz also take the form $0/0$ at the throat. If required, the values of these quantities can be obtained without difficulty by means of the usual method.

6. Viscous swirling flow through a convergent-divergent nozzle

It is first necessary to make assumptions concerning the functions to represent w and v. For the former we take

$$w = w_1\left\{\frac{2(a-r)}{a-b}-\frac{(a-r)^2}{(a-b)^2}\right\}, \tag{6.1}$$

where w_1 is the value of w at the free surface. This expression, which is analogous to (3.6), satisfies the conditions

$$\left.\begin{aligned} w &= 0 \quad \text{at } r = a \\ w &= w_1 \quad \text{at } r = b \\ \frac{\partial w}{\partial r} &= 0 \quad \text{at } r = b \end{aligned}\right\}. \tag{6.2}$$

For v we assume

$$v = \frac{v_1}{b(a-b)^2}\{-ab^2+(a^2+b^2)r-ar^2\}, \tag{6.3}$$

which meets the requirements

$$\left.\begin{aligned} v &= 0 \quad \text{at } r = a \\ v &= v_1 \quad \text{at } r = b \\ \frac{\partial v}{\partial r}-\frac{v}{r} &= 0 \quad \text{at } r = b \end{aligned}\right\}, \tag{6.4}$$

the last condition ensuring zero tangential force at the free surface. For $b/a = \frac{1}{2}$ the numerical values of (6.1) and (6.3) are displayed in Fig. 4. The radial velocity being negligible, the variation of the pressure head p with radius is shown by Goldstein (**5**) to be given by the radial equation of motion

$$\frac{\partial p}{\partial r} = \frac{v^2}{r}, \quad \text{with } p = 0 \text{ at } r = b. \tag{6.5}$$

Writing

$$\delta = a-b, \tag{6.6}$$

we then obtain the following results which are wanted later:

$$Q = \int_b^a rw\,dr = w_1\frac{\delta}{12}(8a-5\delta); \tag{6.7}$$

$$\int_b^a rw^2\,dr = w_1^2\frac{\delta}{30}(16a-11\delta); \tag{6.8}$$

$$\int_b^a r^2vw\,dr = v_1 w_1 \frac{\delta}{420(a-\delta)}(224a^3-483a^2\delta+368a\delta^2-98\delta^3); \qquad (6.9)$$

$$p_2 = \frac{v_1^2}{12\delta^4(a-\delta)^2}\left\{12a^2(a-\delta)^4\log\frac{a}{a-\delta}+\delta(-12a^5+42a^4\delta-\right.$$
$$\left.-52a^3\delta^2+25a^2\delta^3+4a\delta^4-6\delta^5)\right\}, \qquad (6.10)$$

p_2 being the pressure head at the wall where $r = a$;

$$\int_b^a rp\,dr = \frac{v_1^2}{120\delta^4(a-\delta)^2}\left\{60a^4(a-\delta)^4\log\frac{a}{a-\delta}+\delta(-60a^7+210a^6\delta-\right.$$
$$\left.-260a^5\delta^2+125a^4\delta^3-12a^3\delta^4+42a^2\delta^5-56a\delta^6+15\delta^7)\right\}; \qquad (6.11)$$

$$\left[\frac{\partial w}{\partial r}\right]_{r=a} = -w_1\frac{2}{\delta}; \qquad (6.12)$$

$$\left[\frac{\partial v}{\partial r}\right]_{r=a} = -v_1\frac{2a-\delta}{\delta(a-\delta)}. \qquad (6.13)$$

As a check, it was proved that both (6.10) and (6.11) $\to 0$ as $\delta \to 0$.

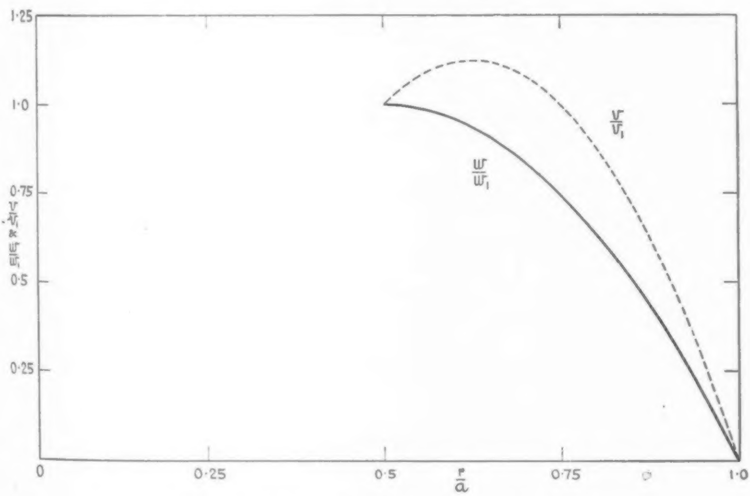

FIG. 4. Assumed velocity distributions in nozzle for $b/a = \frac{1}{2}$.

We have to employ both axial and angular momentum integrals, and they will be derived from the axial and angular equations of motion given

by Goldstein (5), which for axial symmetry reduce to

$$ru\frac{\partial w}{\partial r}+rw\frac{\partial w}{\partial z} = -r\frac{\partial p}{\partial z}+\nu\left(r\frac{\partial^2 w}{\partial r^2}+\frac{\partial w}{\partial r}+r\frac{\partial^2 w}{\partial z^2}\right), \tag{6.14}$$

and
$$r^2u\frac{\partial v}{\partial r}+r^2w\frac{\partial v}{\partial z}+ruv = \nu\left(r^2\frac{\partial^2 v}{\partial r^2}+r\frac{\partial v}{\partial r}-v+r^2\frac{\partial^2 v}{\partial z^2}\right). \tag{6.15}$$

Here we retain the terms involving the radial velocity u; but, since the constriction is gradual, we shall follow the approximations of boundary-layer theory and reject the final $\partial^2/\partial z^2$ terms in both expressions. Goldstein showed also that the equation of continuity is

$$\frac{\partial}{\partial r}(ru)+r\frac{\partial w}{\partial z} = 0. \tag{6.16}$$

Now, from (6.16),

$$\frac{\partial}{\partial r}(ruw) = w\frac{\partial}{\partial r}(ru)+ru\frac{\partial w}{\partial r} = -rw\frac{\partial w}{\partial z}+ru\frac{\partial w}{\partial r}, \tag{6.17}$$

hence
$$ru\frac{\partial w}{\partial r}+rw\frac{\partial w}{\partial z} = \frac{\partial}{\partial r}(ruw)+2rw\frac{\partial w}{\partial z}; \tag{6.18}$$

therefore, if u_1 is the value of u at the free surface,

$$\int\limits_b^a \left(ru\frac{\partial w}{\partial r}+rw\frac{\partial w}{\partial z}\right)dr = -bu_1w_1+\int\limits_b^a \frac{\partial}{\partial z}(rw^2)\,dr, \tag{6.19}$$

since $w = 0$ at $r = a$. Next, we use the theorem that, if $f(r,a,b)$ is a function of r, a, and b,

$$\frac{\partial}{\partial z}\int\limits_b^a f(r,a,b)\,dr = \int\limits_b^a \frac{\partial}{\partial z}f(r,a,b)\,dr+f(a,a,b)\frac{da}{dz}-f(b,a,b)\frac{db}{dz}. \tag{6.20}$$

On applying this to rw^2, we find

$$\frac{\partial}{\partial z}\int\limits_b^a rw^2\,dr = \int\limits_b^a \frac{\partial}{\partial z}(rw^2)\,dr-bw_1^2\frac{db}{dz}, \tag{6.21}$$

again using the condition $w = 0$ at $r = a$. From the geometry of the motion it appears that

$$\frac{u_1}{w_1} = \frac{db}{dz}, \tag{6.22}$$

therefore with the aid of (6.21) the right-hand side of (6.19) reduces to

$$\frac{\partial}{\partial z}\int\limits_b^a rw^2\,dr. \tag{6.23}$$

Again using (6.20), we have

$$\frac{\partial}{\partial z}\int\limits_b^a rp\,dr = \int\limits_b^a \frac{\partial}{\partial z}(rp)\,dr + ap_2\frac{da}{dz}, \tag{6.24}$$

since $p = 0$ at $r = b$. Then, on integrating (6.14) and using (6.23) and (6.24), we obtain the axial momentum integral

$$\frac{\partial}{\partial z}\int\limits_b^a rw^2\,dr = -\frac{\partial}{\partial z}\int\limits_b^a rp\,dr + ap_2\frac{da}{dz} + ra\left[\frac{\partial w}{\partial r}\right]_{r=a}, \tag{6.25}$$

$\partial w/\partial r$ being zero at $r = b$. If we consider two cross-sections of the nozzle a small distance dz apart, we find from first principles that the physical interpretation of (6.25) is as follows: the increase in the flow of axial momentum is equal to minus the increase of axial pressure over the annular section plus the axial component of the wall pressure minus the viscous force due to the wall.

The angular momentum integral will now be derived by a similar process. From (6.16)

$$\frac{\partial}{\partial r}(r^2uv) = rv\frac{\partial}{\partial r}(ru) + ru\frac{\partial}{\partial r}(rv) = -r^2v\frac{\partial w}{\partial z} + r^2u\frac{\partial v}{\partial r} + ruv, \tag{6.26}$$

hence
$$r^2u\frac{\partial v}{\partial r} + r^2w\frac{\partial v}{\partial z} + ruv = \frac{\partial}{\partial r}(r^2uv) + \frac{\partial}{\partial z}(r^2vw), \tag{6.27}$$

therefore

$$\int\limits_b^a \left(r^2u\frac{\partial v}{\partial r} + r^2w\frac{\partial v}{\partial z} + ruv\right)dr = -b^2u_1v_1 + \int\limits_b^a \frac{\partial}{\partial z}(r^2vw)\,dr, \tag{6.28}$$

since v is zero at the wall. From (6.20) and (6.22) we have

$$\frac{\partial}{\partial z}\int\limits_b^a r^2vw\,dr = \int\limits_b^a \frac{\partial}{\partial z}(r^2vw)\,dr - b^2u_1v_1, \tag{6.29}$$

again using the condition that $v = 0$ at $r = a$. Thus the right-hand side of (6.28) is simply

$$\frac{\partial}{\partial z}\int\limits_b^a r^2vw\,dr. \tag{6.30}$$

Now

$$\int_b^a \left(r^2\frac{\partial^2 v}{\partial r^2}+r\frac{\partial v}{\partial r}-v\right)dr = \left[r^2\frac{\partial v}{\partial r}-rv\right]_b^a = a^2\left[\frac{\partial v}{\partial r}\right]_{r=a}, \tag{6.31}$$

because at $r = a$, $v = 0$; and at $r = b$, $\partial v/\partial r = v/r$ in conformity with (6.4). Then, on integrating (6.15), we arrive with the help of (6.30) and (6.31) at the angular momentum integral

$$\frac{\partial}{\partial z}\int_b^a r^2vw\,dr = \nu a^2\left[\frac{\partial v}{\partial r}\right]_{r=a}. \tag{6.32}$$

Again, this may be expressed in words as: the increase in the flow of angular momentum is equal to minus the viscous couple exerted by the wall. The term omitted in (6.15) evidently represents the difference between the viscous couples exerted on the annular ends of the element by the neighbouring layers.

The formulae (6.7)–(6.13) are now inserted in the momentum integrals. First, w_1 is eliminated from (6.8), (6.9), and (6.12) by means of (6.7), so that the variables, with which we shall be concerned, are δ, a, and v_1. To obtain results in non-dimensional form, we again use (5.4) and (5.5), and add

$$L = \frac{\nu a}{Q}. \tag{6.33}$$

For the angular momentum integral, (6.9) and (6.13) are required, from which, after performing the necessary differentiation, we obtain without further approximation

$$\mathbf{B}\frac{d\delta}{dz}-\frac{a}{v_1}\frac{dv_1}{dz} = \mathbf{A}\frac{da}{dz}+\mathbf{C}L, \tag{6.34}$$

where $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, which are functions of R only, are given by

$$\mathbf{A} = \frac{1792-5824R+6695R^2-3262R^3+566R^4}{(8-5R)(1-R)(224-483R+368R^2-98R^3)}, \tag{6.35}$$

$$\mathbf{B} = \frac{952-3648R+4721R^2-2548R^3+490R^4}{(8-5R)(1-R)(224-483R+368R^2-98R^3)}, \tag{6.36}$$

$$\mathbf{C} = \frac{35(8-5R)(2-R)}{R(224-483R+368R^2-98R^3)}. \tag{6.37}$$

The axial momentum integral involves (6.8) and (6.10)–(6.12), and from them after very considerable labour we find

$$(\mathbf{E}K-\mathbf{H})\frac{d\delta}{dz}-\mathbf{F}\frac{a}{v_1}\frac{dv_1}{dz} = (\mathbf{G}-\mathbf{D}K)\frac{da}{dz}+KL, \tag{6.38}$$

in which

$$\mathbf{D} = \frac{32R(4-3R)}{5(8-5R)^2}, \tag{6.39}$$

$$\mathbf{E} = \frac{2(64-120R+55R^2)}{5(8-5R)^2}, \tag{6.40}$$

$$\mathbf{F} = \frac{8-5R}{1440R^2(1-R)^2}\{-60(1-R)^4\log(1-R)+R(-60+210R- \\ -260R^2+125R^3-12R^4+42R^5-56R^6+15R^7)\}, \tag{6.41}$$

$$\mathbf{G} = \frac{8-5R}{1440R^2(1-R)^3}\{-60(1-R)^4(2-R)\log(1-R)+R(-120+480R- \\ -730R^2+510R^3-181R^4+68R^5-44R^6+13R^7)\}, \tag{6.42}$$

$$\mathbf{H} = \frac{8-5R}{1440R^3(1-R)^3}\{60(1-R)^4(2-R)\log(1-R)+R(120-480R+ \\ +730R^2-510R^3+149R^2+36R^5-84R^6+58R^7-15R^8)\}. \tag{6.43}$$

Thus we have two simultaneous equations (6.34) and (6.35) for $d\delta/dz$ and dv_1/dz. The results of solving them are

$$\frac{d\delta}{dz} = -\frac{\{\mathbf{D}K+\mathbf{AF}-\mathbf{G}\}(da/dz)+\{\mathbf{CF}-K\}L}{\mathbf{E}K-(\mathbf{BF}+\mathbf{H})}, \tag{6.44}$$

$$\frac{a}{v_1}\frac{dv_1}{dz} = -\frac{\{(\mathbf{AE}+\mathbf{BD})K-(\mathbf{AH}+\mathbf{BG})\}(da/dz)-\{(\mathbf{B}-\mathbf{CE})K+\mathbf{CH}\}L}{\mathbf{E}K-(\mathbf{BF}+\mathbf{H})}; \tag{6.45}$$

they should be compared with (5.8) and (5.9), which are the corresponding results for inviscid flow. Again, $d\delta/dz$ must take the form 0/0 at the effective throat, the position of which is therefore given by the two conditions

$$\mathbf{E}K-(\mathbf{BF}+\mathbf{H}) = 0, \tag{6.46}$$

$$\{\mathbf{D}K+\mathbf{AF}-\mathbf{G}\}\frac{da}{dz}+\{\mathbf{CF}-K\}L = 0. \tag{6.47}$$

On inserting (6.46) into the numerators of (6.44) and (6.45), we find the latter to be equal to the former multiplied by **B**, without reference to (6.35)–(6.37) and (6.39)–(6.43). Hence dv_1/dz also takes the form 0/0 at the effective throat. The result (6.44) is so complicated that it is impracticable to derive an expression for the throat value of $d\delta/dz$ by the usual method of differentiation.

It will now be proved that a long wave of small amplitude can maintain itself stationary on the streaming and revolving surface of the liquid at the effective throat. As shown in Fig. 5, we suppose a stationary wave to exist having an elevation ϵ at the section XY so that the surface there is

at radius $b-\epsilon$, and we denote by w' and v' the small additional axial and tangential velocities at Y due to the wave. In conformity with (6.1) and (6.3), the distributions of axial and tangential velocities across XY are taken to be

$$w = (w_1+w')\left\{\frac{2(a-r)}{a-(b-\epsilon)}-\frac{(a-r)^2}{\{a-(b-\epsilon)\}^2}\right\}, \tag{6.48}$$

$$v = \frac{v_1+v'}{(b-\epsilon)\{a-(b-\epsilon)\}^2}[-a(b-\epsilon)^2+\{a^2+(b-\epsilon)^2\}r-ar^2]. \tag{6.49}$$

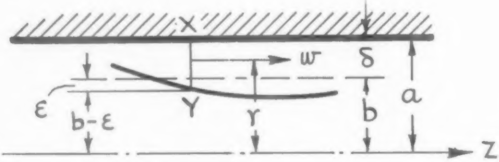

FIG. 5. Wave on free surface in nozzle.

A number of integrals across XY are required, and they can be obtained from (6.7), (6.8), (6.9), and (6.11) on replacing b by $b-\epsilon$, δ by $\delta+\epsilon$, w_1 by w_1+w', and v_1 by v_1+v'. Thus with the substitutions (6.48) and (6.49) we obtain

$$Q = \int\limits_{b-\epsilon}^{a} rw\,dr = (w_1+w')\left(\frac{\delta+\epsilon}{12}\right)\{8a-5(\delta+\epsilon)\}, \tag{6.50}$$

$$\int\limits_{b-\epsilon}^{a} rw^2\,dr = \frac{24}{5}Q^2\frac{16a-11(\delta+\epsilon)}{(\delta+\epsilon)\{8a-5(\delta+\epsilon)\}^2}, \tag{6.51}$$

$$\int\limits_{b-\epsilon}^{a} r^2vw\,dr = \frac{Q}{35}(v_1+v')\frac{\{224a^3-483a^2(\delta+\epsilon)+368a\delta(\delta+2\epsilon)-98\delta^2(\delta+3\epsilon)\}}{\{8a-5(\delta+\epsilon)\}\{a-(\delta+\epsilon)\}}, \tag{6.52}$$

$$\begin{aligned}\int\limits_{b-\epsilon}^{a} rp\,dr = \frac{(v_1+v')^2}{120\delta^3(\delta+4\epsilon)\{a-(\delta+\epsilon)\}^2}\Big[&60a^4\{a-(\delta+\epsilon)\}^4\log\left\{\frac{a}{a-(\delta+\epsilon)}\right\}-\\ &-60a^7(\delta+\epsilon)+210a^6\delta(\delta+2\epsilon)-260a^5\delta^2(\delta+3\epsilon)+\\ &+125a^4\delta^3(\delta+4\epsilon)-12a^3\delta^4(\delta+5\epsilon)+42a^2\delta^5(\delta+6\epsilon)-\\ &-56a\delta^6(\delta+7\epsilon)+15\delta^7(\delta+8\epsilon)\Big].\end{aligned} \tag{6.53}$$

In (6.51) and (6.52) w_1+w' has been eliminated by means of (6.50). The axial momentum equation (6.25) is now

$$\frac{\partial}{\partial z}\int\limits_{b-\epsilon}^{a} rw^2\,dr = -\frac{\partial}{\partial z}\int\limits_{b-\epsilon}^{a} rp\,dr. \tag{6.54}$$

We therefore differentiate (6.51) and (6.53), remembering that v' and ϵ

are variables and that v_1, a, and δ are constants. After protracted calculations, in which squares and products of small quantities are again ignored, (6.54) becomes

$$-\frac{24Q^2\mathbf{E}}{a^3R^2(8-5R)}\frac{d\epsilon}{dz} = -\frac{24av_1^2\mathbf{H}}{R^2(8-5R)}\frac{d\epsilon}{dz} - \frac{24a^2v_1\mathbf{F}}{R^2(8-5R)}\frac{dv'}{dz}, \tag{6.55}$$

and this reduces to $$(\mathbf{E}K-\mathbf{H})\frac{d\epsilon}{dz} = \mathbf{F}\frac{a}{v_1}\frac{dv'}{dz}. \tag{6.56}$$

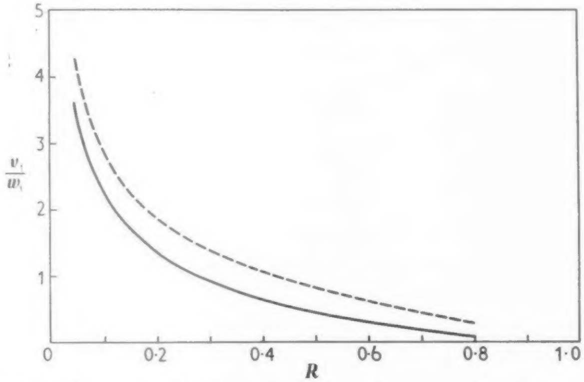

FIG. 6. Values of v_1/w_1 at effective throat of nozzle: —— viscous, - - - - inviscid.

The angular momentum equation (6.32) in this instance is

$$\frac{\partial}{\partial z}\int\limits_{b-\epsilon}^{a} r^2vw\,dr = 0, \tag{6.57}$$

which, after (6.52) has been differentiated, yields

$$\frac{dv'}{dz} - \mathbf{B}\frac{v_1}{a}\frac{d\epsilon}{dz} = 0. \tag{6.58}$$

Hence with (6.56) we arrive at the condition

$$\mathbf{E}K-(\mathbf{BF}+\mathbf{H}) = 0, \tag{6.59}$$

in agreement with (6.46). There is, of course, no result corresponding to (6.47), which involves ν and da/dz.

The relations that exist at the effective throat may be stated in another manner. Equations (6.46) and (6.7) are expressions for $Q^2/(v_1^2a^4)$ and $Q^2/(w_1^2a^4)$, hence by division the interesting ratio v_1/w_1 is given by

$$\frac{v_1}{w_1} = \frac{R}{12}(8-5R)\left\{\frac{\mathbf{E}}{\mathbf{BF}+\mathbf{H}}\right\}^{\frac{1}{2}}. \tag{6.60}$$

This ratio has been plotted on a base of R in Fig. 6, where it is shown by

the full line. The combination of (5.1) and (5.10) yields the corresponding result for frictionless flow which is

$$\frac{v_1}{w} = \left\{\frac{2(1-R)^2}{R(2-R)}\right\}^{\frac{1}{2}}, \tag{6.61}$$

and this is displayed on the same diagram by a broken line.

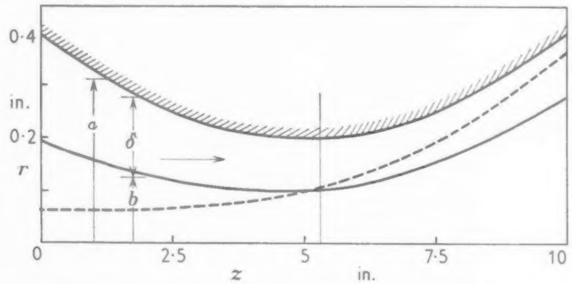

FIG. 7. Profiles of nozzle and stream: —— viscous, - - - - inviscid.

7. Numerical example of swirling flow through a convergent-divergent nozzle

The nozzle is taken to be of the symmetrical parabolic shape defined by

$$a = 0{\cdot}4 - 0{\cdot}08z\left(1 - \frac{z}{10}\right). \tag{7.1}$$

It extends from $z = 0$ to $z = 10$; and at the throat $a = 0{\cdot}2$. This shape accords with the requirement that the constriction should be gradual; it is shown to very unequal scales in Fig. 7. The conditions at the effective throat are assumed to be $z = 5{\cdot}3$, $R = 0{\cdot}5$, and these lead to a negative value of $d\delta/dz$ which is wanted in order that comparison may be made with the free flow of inviscid liquid. These conditions were not chosen without some preliminary trials. The value of $d\delta/dz$ is obtained from a quadratic equation analogous to (3.15), both roots of which may be positive if the viscous forces are made excessive by the selection of too small a value of R. For the above values of z and R it follows from (6.46) and (6.47) that $K = 0{\cdot}255327$, $L = 0{\cdot}00821695$. If we suppose (7.1) to be expressed in inches and take the liquid to be an oil for which $\nu = 57{\cdot}6 \times 10^{-3}$ in.^{2}/sec., we find from (6.33) that $Q = 1{\cdot}407026$ in.^{3}/sec.

The solution was very kindly obtained by Dr. J. C. P. Miller and Dr. D. J. Wheeler. The two main difficulties were due to the 0/0 form of the equation for $d\delta/dz$ at the effective throat. The first difficulty arose in evaluating the numerical values of the coefficients of the quadratic

equation mentioned above. These depended on derivatives of very complicated expressions which it was not feasible to differentiate algebraically in full. Instead, numerical derivatives were computed for each intermediate function used in the evaluation of the expression; that is, algebraic substitution was not carried out completely before numerical evaluation was started. As a simple example, if the derivative of

$$y = (x^3+3x)\log x$$

was needed, the expression

$$\frac{dy}{dx} = (3x^2+3)\log x+(x^2+3)$$

would never have been obtained; instead, numerical values would have been found for

$$u = x^3+3x, \qquad u' = 3x^2+3, \qquad v = \log x, \qquad v' = \frac{1}{x};$$

and then $y' = uv'+u'v$ would have been evaluated. Though the variation in method is trivial in this example, the alternation of computational and algebraic processes in this way made a considerable difference to the ease with which computations were carried out.

The second difficulty arose in the evaluation of the solution to

$$d\delta/dz = N/D, \text{ say,}$$

in the neighbourhood of the effective throat, where $N = D = 0$, so that it was very difficult to obtain an accurate value for the quotient. This was overcome by solving two extra differential equations in the neighbourhood of the throat, that is $dN/dz = N'$, $dD/dz = D'$, where N' and D' were computed as outlined above, and by using these equations to give N and D to the desired accuracy.

The calculations were performed by EDSAC. The machine provided values of R and K at small intervals of z in both directions along the nozzle, and by means of (5.4), (5.5), and (6.7) the values of δ, v_1, and w_1 were obtained that are shown in Figs. 7 and 8 by full lines. For comparison, the broken lines in these figures give the corresponding quantities for inviscid flow on the assumption that the discharge and the core radius at the centre of the nozzle are the same as before.

In this example the effects of viscosity are large. With inviscid flow the thickness δ so rapidly diminishes from inlet to outlet that the core radius b increases throughout the nozzle. In viscous flow the reduction in δ is much less marked, and near the outlet, over the range $8{\cdot}8 < z < 10$, the opposing forces are sufficiently great to cause a very slight increase in δ, which is hardly noticeable in the diagram. The changes in thickness are so small

that in the convergent part the core radius decreases to a minimum near $z = 4{\cdot}4$. Fig. 8 shows how the inviscid values of the tangential surface velocity v_1 vary through the nozzle inversely with the core radius. In viscous flow a larger value of v_1 must be imposed at inlet; the reduction is

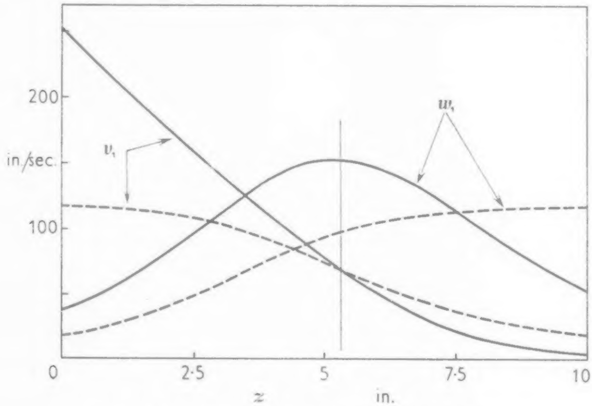

FIG. 8. Component velocities at free surface in nozzle: —— viscous, - - - - inviscid.

now sharper, and this component of velocity is almost extinguished when the outlet is reached. The streaming velocity w_1 in inviscid flow, which is uniform over the cross-section, increases throughout, but in viscous flow it reaches a maximum near $z = 5{\cdot}2$ and thereafter diminishes.

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A NOTE ON THE OSEEN APPROXIMATION FOR A PARABOLOID IN A UNIFORM STREAM PARALLEL TO ITS AXIS

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SUMMARY

It is shown in this note that the Oseen equations have a simple solution when the body is in the form of a paraboloid of elliptic section in a stream of viscous, incompressible fluid which at infinity has uniform velocity in the direction of the axis. Particular attention is given to the case in which the paraboloid degenerates into a flat plate with a parabolic leading edge, and it is suggested that the solution of the Oseen equations in this case may throw some light on the corresponding three-dimensional boundary-layer problem.

1. Introduction

CARRIER and Lewis (**1**) first gave the solution of the Oseen equations for the half-plane $z = 0$, $x > 0$ in a uniform stream. At about the same time Stewartson, working independently, produced the same solution by a particularly simple method which also gave the solution in the case of a parabolic cylinder. Stewartson did not publish this result. A recent paper by Kaplun (**2**) contains a solution similar to Stewartson's.

For the parabolic cylinder the solution of the Oseen equations can be put in the form

$$u = \frac{\partial\phi}{\partial x}+\frac{1}{2k}\frac{\partial\chi}{\partial x}-\chi, \qquad v = 0, \qquad w = \frac{\partial\phi}{\partial z}+\frac{1}{2k}\frac{\partial\chi}{\partial z}, \tag{1}$$

where $$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial z^2} = 0, \qquad \frac{\partial^2\chi}{\partial x^2}+\frac{\partial^2\chi}{\partial z^2} = 2k\frac{\partial\chi}{\partial x}, \qquad k = U/2\nu,$$

and u, $w \to 0$ at infinity and $u = -U$, $w = 0$ on the cylinder.

Stewartson expresses these equations in terms of parabolic cylinder coordinates ξ, η defined by $x = \xi^2-\eta^2$, $z = 2\xi\eta$ in which the cylinder is the coordinate line $\eta =$ constant. He then seeks, and finds, functions ϕ, χ which are functions of η only.

The question naturally arises: Can this method be extended from the parabolic cylinder in two dimensions to the paraboloid in three dimensions by the use of paraboloidal coordinates? It is shown below that this extension

is possible for paraboloids of elliptic section with vertices facing into the uniform stream.

2. The equations

Paraboloidal coordinates (ξ, η, ζ) can be defined as the roots λ of the equation

$$\frac{y^2}{a-\lambda}+\frac{z^2}{b-\lambda} = 4(x-\lambda) \tag{2}$$

when x, y, z are supposed given, $a > b \geqslant 0$ and $\xi \geqslant a$, $a \geqslant \eta \geqslant b$, $b \geqslant \zeta$.

Then the surface $\zeta = \zeta_0$ is the paraboloid of elliptic section

$$\frac{y^2}{a-\zeta_0}+\frac{z^2}{b-\zeta_0} = 4(x-\zeta_0) \tag{3}$$

and by a suitable choice of coordinates in the first place any such paraboloid, except that with a circular section, can be expressed as $\zeta = \zeta_0$ if a, b and ζ_0 are correctly chosen.

The coordinates x, y, z are given in terms of ξ, η, ζ by

$$\begin{aligned} x &= \xi+\eta+\zeta-a-b, \\ y^2 &= \frac{4}{(a-b)}(\xi-a)(a-\eta)(a-\zeta), \\ z^2 &= \frac{4}{(a-b)}(\xi-b)(\eta-b)(b-\zeta). \end{aligned} \tag{4}$$

If the flow has velocity U in the direction of the x-axis of infinity, the Oseen equations are

$$\begin{aligned} U\frac{\partial u}{\partial x} &= -\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right), \\ U\frac{\partial v}{\partial x} &= -\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right), \\ U\frac{\partial w}{\partial x} &= -\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right), \\ 0 &= \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} \end{aligned} \tag{5}$$

with $u = -U$, $v = w = 0$ on $\zeta = \zeta_0$ and $u, v, w \to 0$ at infinity.

In the general three-dimensional problem it is not possible to write the solution of the equations (5) in a form corresponding to equations (1), but

in this case a solution is to be sought of the form

$$\left.\begin{aligned} u &= \frac{\partial\phi}{\partial x}+\frac{1}{2k}\frac{\partial\chi}{\partial x}-\chi, \\ v &= \frac{\partial\phi}{\partial y}+\frac{1}{2k}\frac{\partial\chi}{\partial y}, \\ w &= \frac{\partial\phi}{\partial z}+\frac{1}{2k}\frac{\partial\chi}{\partial z}, \\ p &= -\rho U\frac{\partial\phi}{\partial x}, \end{aligned}\right\} \tag{6}$$

where
$$\nabla^2\phi \equiv \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2} = 0, \tag{7}$$

$$\nabla^2\chi \equiv \frac{\partial^2\chi}{\partial x^2}+\frac{\partial^2\chi}{\partial y^2}+\frac{\partial^2\chi}{\partial z^2} = 2k\frac{\partial\chi}{\partial x}, \tag{8}$$

where $k = U/2\nu$ and in paraboloidal coordinates

$$\begin{aligned}\nabla^2 \equiv {} & \frac{\{(\xi-a)(\xi-b)\}^{\frac{1}{2}}}{(\xi-\zeta)(\xi-\eta)}\frac{\partial}{\partial\xi}\left[\{(\xi-a)(\xi-b)\}^{\frac{1}{2}}\frac{\partial}{\partial\xi}\right]+ \\ & +\frac{\{(a-\eta)(\eta-b)\}^{\frac{1}{2}}}{(\eta-\zeta)(\xi-\eta)}\frac{\partial}{\partial\eta}\left[\{(a-\eta)(\eta-b)\}^{\frac{1}{2}}\frac{\partial}{\partial\eta}\right]+ \\ & +\frac{\{(a-\zeta)(b-\zeta)\}^{\frac{1}{2}}}{(\eta-\zeta)(\xi-\zeta)}\frac{\partial}{\partial\zeta}\left[\{(a-\zeta)(b-\zeta)\}^{\frac{1}{2}}\frac{\partial}{\partial\zeta}\right]\end{aligned}$$

and
$$\frac{\partial}{\partial x} \equiv \frac{(\xi-a)(\xi-b)}{(\xi-\zeta)(\xi-\eta)}\frac{\partial}{\partial\xi}+\frac{(a-\eta)(\eta-b)}{(\xi-\eta)(\eta-\zeta)}\frac{\partial}{\partial\eta}+\frac{(a-\zeta)(b-\zeta)}{(\eta-\zeta)(\xi-\zeta)}\frac{\partial}{\partial\zeta}.$$

The existence of such a set of solutions implies that the vorticity always lies in a plane perpendicular to the axis of the paraboloid. Moreover, the form of equations (7) and (8) in paraboloidal coordinates makes it possible to find solutions of these equations which are functions of ζ only.

In this case (7) and (8) become

$$\frac{d}{d\zeta}\left[\{(a-\zeta)(b-\zeta)\}^{\frac{1}{2}}\frac{d\phi}{d\zeta}\right] = 0, \tag{9}$$

$$\frac{d}{d\zeta}\left[\{(a-\zeta)(b-\zeta)\}^{\frac{1}{2}}\frac{d\chi}{d\zeta}\right] = 2k\{(a-\zeta)(b-\zeta)\}^{\frac{1}{2}}\frac{d\chi}{d\zeta}. \tag{10}$$

Solutions of these equations may be written as

$$\phi = A\log[a/2+b/2-\zeta+\{(a-\zeta)(b-\zeta)\}^{\frac{1}{2}}], \tag{11}$$

$$\chi = B\int\limits_{-\infty}^{\zeta}\frac{e^{2ks}}{\{(a-s)(b-s)\}^{\frac{1}{2}}}\,ds, \tag{12}$$

where A and B are constants.

Equations (6) can now be written as

$$\begin{aligned}
u &= \frac{(a-\zeta)(b-\zeta)}{(\xi-\zeta)(\eta-\zeta)}\frac{d}{d\zeta}\left(\phi+\frac{1}{2k}\chi\right)-\chi \\
&= \frac{\{(a-\zeta)(b-\zeta)\}^{\frac{1}{2}}}{(\xi-\zeta)(\eta-\zeta)}\left\{A+\frac{Be^{2k\zeta}}{2k}\right\}-\chi, \\
v &= \frac{-(b-\zeta)\{(\xi-a)(a-\eta)(a-\zeta)\}^{\frac{1}{2}}}{(a-b)^{\frac{1}{2}}(\xi-\zeta)(\eta-\zeta)}\frac{d}{d\zeta}\left(\phi+\frac{1}{2k}\chi\right) \\
&= \frac{-\{(\xi-a)(a-\eta)(b-\zeta)\}^{\frac{1}{2}}}{(a-b)^{\frac{1}{2}}(\xi-\zeta)(\eta-\zeta)}\left\{A+\frac{Be^{2k\zeta}}{2k}\right\}, \\
w &= \frac{-(a-\zeta)\{(\xi-b)(\eta-b)(b-\zeta)\}^{\frac{1}{2}}}{(a-b)^{\frac{1}{2}}(\xi-\zeta)(\eta-\zeta)}\frac{d}{d\zeta}\left(\phi+\frac{1}{2k}\chi\right) \\
&= \frac{-\{(\xi-b)(\eta-b)(a-\zeta)\}^{\frac{1}{2}}}{(a-b)^{\frac{1}{2}}(\xi-\zeta)(\eta-\zeta)}\left\{A+\frac{Be^{2k\zeta}}{2k}\right\}.
\end{aligned} \tag{13}$$

The conditions $v = w = 0$ for $\zeta = \zeta_0$ are satisfied by taking $A = -\frac{Be^{2k\zeta_0}}{2k}$ and the condition $u = -U$ is satisfied by taking

$$\chi(\zeta_0) = B\int\limits_{-\infty}^{\zeta_0}\frac{e^{2ks}}{\{(a-s)(b-s)\}^{\frac{1}{2}}}\,ds = U. \tag{14}$$

It is clear that $u, v, w \to 0$ as $\zeta \to -\infty$ which corresponds to moving off to infinite distance from the solid, and it is not difficult to show that the vortex lines are the plane ellipses

$$\zeta = \text{constant}, \qquad x = \text{constant}.$$

Equation (3) cannot represent a paraboloid of circular section unless $a = b$, when the coordinates become degenerate and the method fails. However, a result due to Howarth may be employed in this case. The coordinates ξ, η are introduced, where

$$x = \xi^2-\eta^2, \qquad y^2+z^2 = 4\xi^2\eta^2.$$

A solution of the type given in equations (6) can then be found with ϕ and χ functions of η only. The equations corresponding to (9) and (10) are the equations

$$\frac{d}{d\eta}\left(\eta\frac{d\phi}{d\eta}\right) = 0 \tag{15}$$

and

$$\frac{d}{d\eta}\left(\eta\frac{d\chi}{d\eta}\right) = -4k\eta^2\frac{d\chi}{d\eta}, \tag{16}$$

with solutions $\qquad \phi = A\log\eta, \qquad \chi = B\int\limits_{\eta}^{\infty}\frac{e^{-2ks^2}}{s}\,ds. \qquad (17)$

The boundary conditions can then be satisfied by taking $2kA = Be^{-2k\eta_0^2}$,

$$B\int_{\eta_0}^{\infty} \frac{e^{-2ks^2}}{s}\,ds = U,$$

where the paraboloid of revolution is the surface $\eta = \eta_0$.

The solution of the Oseen equations specified in equations (11), (12), (13) and (14), will in particular apply to the degenerate paraboloid $\zeta = \zeta_0 = b$, which is a flat plate with a parabolic leading edge. In dealing with this case it is convenient to take $b = \zeta_0 = 0$, when the parabolic plate is $z = 0$, $4ax > y^2$.

Then
$$B = U\left(\int_{-\infty}^{0} \frac{e^{2ks}\,ds}{\{-s(a-s)\}^{\frac{1}{2}}}\right)^{-1} = \frac{Ue^{-ka}}{K_0(ka)} = \frac{Ue^{-R/4}}{K_0(R/4)},$$

where R is a Reynolds number defined as

$$R = \frac{2aU}{\nu} = 4ak,$$

$2a$ being the semi-latus rectum of the parabola.

The fluid velocity $\mathbf{u}$ given by the approximation may then be written

$$\mathbf{u} = \mathbf{U}+\mathbf{u}_1+\mathbf{u}_2,$$

where $\mathbf{U} = (U, 0, 0)$, $\mathbf{u}_1 = (-\chi, 0, 0)$, and $\mathbf{u}_2$ is a vector of magnitude $\dfrac{2aUe^{-R/4}}{RK_0(R/4)}\dfrac{(1-e^{-2k\zeta})}{\{(\xi-\zeta)(\eta-\zeta)\}^{\frac{1}{2}}}$ directed in the ζ-direction—i.e. along the tangent to the curve $\xi =$ constant, $\eta =$ constant in the direction of ζ increasing. The vector $\mathbf{u}_1$ influences only the x-component of the flow and is a pure boundary layer term. As R increases it is significant in a smaller and smaller region round the plate. The secondary flow—the three-dimensional effect induced by the presence of the plate—is given by the term $\mathbf{u}_2$. The magnitude of this term is zero on the plate and at infinity and decreases like $1/R^{\frac{1}{2}}$ as $R \to \infty$. The direction of $\mathbf{u}_2$ is always away from the plate and in the limit, as the plate is approached, is normal to the plate.

This secondary velocity $\mathbf{u}_2$ may be considered as due to a source distribution of density $\dfrac{Ue^{-R/4}e^{-2k\zeta}\{-\zeta(a-\zeta)\}^{\frac{1}{2}}}{K_0(R/4)\{(\xi-\zeta)(\eta-\zeta)\}^{\frac{1}{2}}}$ per unit volume; and this density is always positive.

Again, in this case it is a simple matter to show that on the plate

$$\left(\frac{\partial u}{\partial z}\right)_{z=0} = \frac{B}{\xi\eta}, \qquad \left(\frac{\partial v}{\partial z}\right)_{z=0} = 0$$

so that, although the flow is strictly three-dimensional, the skin friction is in the direction of the flow at infinity. Moreover, when $\zeta = 0$ equations

(4) can be used to show that $\xi\eta = \frac{1}{4}(4ax-y^2)$ so that

$$\left(\frac{\partial u}{\partial z}\right)_{z=0} = \frac{2B}{(4ax-y^2)^{\frac{1}{2}}} = \frac{Ue^{-R/4}}{a^{\frac{1}{2}}\bar{x}^{\frac{1}{2}}K_0(R/4)}, \tag{18}$$

where $\bar{x} = (x-y^2/4a)$ = distance from edge of plate measured parallel to the x-axis.

It is interesting to compare this result with the values of $(\partial u/\partial z)_{z=0}$ obtained from other methods of approximation. If the boundary-layer equations are taken in the form

$$\begin{aligned} u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z} &= \nu\frac{\partial^2 u}{\partial z^2}, \\ u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z} &= \nu\frac{\partial^2 v}{\partial z^2}, \\ u\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} &= 0, \end{aligned} \tag{19}$$

Moore (**3**) has shown that there will be a solution with $v = 0$ and on the plate

$$\left(\frac{\partial u}{\partial z}\right)_{z=0} = \frac{0{\cdot}332UR^{\frac{1}{2}}}{2^{\frac{1}{2}}a^{\frac{1}{2}}\bar{x}^{\frac{1}{2}}}. \tag{20}$$

The ratio of this value to the value given by equation (18) is independent of position on the plate.

However, the simplest approximation to the skin friction on the plate is obtained by taking Rayleigh's value for the skin friction (**4**), starting the solution from the leading edge of the plate. The value of $(\partial u/\partial z)_{z=0}$ obtained is

$$\left(\frac{\partial u}{\partial z}\right)_{z=0} = \frac{UR^{\frac{1}{2}}}{2^{\frac{1}{2}}\pi^{\frac{1}{2}}a^{\frac{1}{2}}\bar{x}^{\frac{1}{2}}}. \tag{21}$$

The ratio of this value to the value obtained from the Oseen equations in equation (18) is independent of position on the plate and is a function only of the Reynolds number R. The way the ratio varies with R is shown in Fig. 1. Beginning at zero for $R = 0$ it is greater than 0·9 when $R = 4$ and rapidly approaches its limiting value of unity. Thus, except for very small values of the Reynolds number, the Oseen equations provide almost no information about the skin friction that cannot be obtained far more easily from this simple Rayleigh approximation. Nor is this result altogether surprising, for the equations which determine the Rayleigh approximation are obtained by forming boundary-layer equations like equations (19) *from the Oseen equations* instead of the full equations of motion. What is of interest is the fact that the omission of the $\partial^2/\partial y^2$ terms from these equations should have so little effect at large distances from the axis of the parabola: for it would be expected that far from the axis, where the direction of the

tangent to the leading edge approaches the direction of the axis, 'edge effects' would become important and the terms in $\partial^2/\partial y^2$ would not be negligible in comparison with the $\partial^2/\partial z^2$ terms.

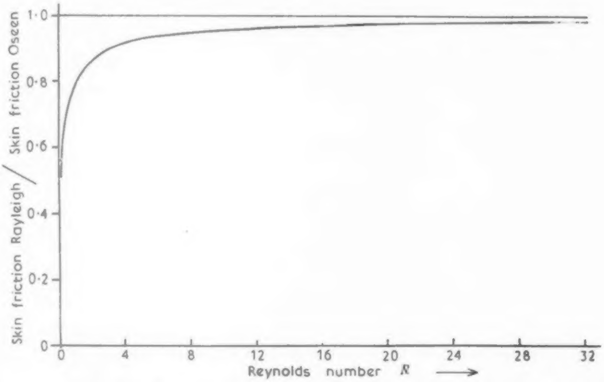

FIG. 1. Ratio of skin friction from Rayleigh approximation to skin friction from Oseen equations.

The fact that edge effects are of so little significance in this approximation suggests that equations (19) may give a valid approximation to the skin friction over the whole plate. Equations (19) ignore edge effects, but as they bear the same relationship to the full equations of motion as the equations determining the Rayleigh approximation do to the Oseen equations, it seems reasonable to suppose that edge effects will have very little effect on skin friction for high Reynolds numbers. If this is the case, equation (20) should give a much better approximation to the skin friction than equation (18) for high Reynolds numbers, for the limitations of the Oseen equations are well known.

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THE INITIAL VALUE PROBLEM FOR THE WAVE EQUATION IN THE DISTRIBUTIONS OF SCHWARTZ

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SUMMARY

The classic formulae for the solution of initial value problems for the wave equation are found in the case of distributions by a Fourier transform method.

1. Summary of classic cases and results

THE following are the classic formulae for the solution of the homogeneous wave equation

$$L(u) \equiv \Delta u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0, \tag{A}$$

with initial conditions

$$u(x,0) = u_0(x), \qquad \frac{\partial}{\partial t}u(x,0) = u_1(x),$$

where Δ is the Laplacian operator $\sum_{i=1}^{k} \frac{\partial^2}{\partial x_i^2}$ in k-dimensional space and x is the point $(x_1, x_2,..., x_k)$:

(i) the case $k = 1$ (equation of vibrating strings), d'Alembert's solution,

$$u(x,t) = \tfrac{1}{2}[u_0(x-ct)+u_0(x+ct)]+\frac{1}{2c}\int\limits_{x-ct}^{x+ct} u_1(\xi)\, d\xi; \tag{1}$$

(ii) the case $k = 3$ (equation of spherical waves), Poisson's solution,

$$u(x,y,z,t) = \frac{d}{dt}[tM_{ct}(u_0)]+tM_{ct}(u_1), \tag{2}$$

where $M_r(u)$ denotes the average value of u over the surface of the sphere of radius r with centre at (x, y, z);

(iii) deduced from this for the case $k = 2$ (equation of cylindrical waves), the Poisson–Parseval solution,

$$u(x,y,t) = \frac{1}{2\pi c}\left[\frac{d}{dt}\mu_{ct}(u_0)+\mu_{ct}(u_1)\right], \tag{3}$$

where $$\mu_r(u) = \iint\limits_{(x-\xi)^2+(y-\eta)^2<r^2} \frac{u(\xi,\eta)}{\sqrt{\{r^2-(x-\xi)^2-(y-\eta)^2\}}}\, d\xi d\eta.$$

The original derivation of Poisson's solution consisted in verifying directly that the formula (2) satisfies every required condition; a theorem of Holmgren then proves that the solution is unique (see **1**, 47).

(iv) the case $c = 1$, $k = 2m+3$, Tedone (**2**) found a formula equivalent to that found later by Hadamard (**1**, 244) as a particular case of his 'finite part' solution of the general non-parabolic partial differential equation of the second order,

$$\frac{4\pi^k}{\omega_k} u(x,t) = \frac{d}{dt}\left(\frac{d}{2t\,dt}\right)^{k-3}[t^{2k-3}M_t(u_0)]+\left(\frac{d}{2t\,dt}\right)^{k-3}[t^{2k-3}M_t(u_1)],$$

where ω_k is the area of the unit hypersphere in k-dimensional space and $M_r(u)$ denotes the average value of u over the hypersphere of radius r with centre at x. Hadamard's method also furnishes a direct proof of the existence of Poisson's solution. The same solution was found in the alternative form

$$u(x,t) = \frac{1}{(k-2)!}\left\{\left(\frac{\partial}{\partial t}\right)^{k-3}\int\limits_0^t (t^2-r^2)^{\frac{1}{2}(k-3)}rM_r(u_0)\,dr + \right.$$
$$\left. +\left(\frac{\partial}{\partial t}\right)^{k-2}\int\limits_0^t (t^2-r^2)^{\frac{1}{2}(k-3)}rM_r(u_1)\,dr\right\} \quad (4)$$

by Courant (**3**, 387) by a series of formal operations on multiple Fourier transforms, the resulting formula being verified *a posteriori* as satisfying all the required conditions. From this solution he then deduced by Hadamard's method of descent

(v) the same formula (4) in the case $k = 2m+2$.

Courant's method was subsequently made rigorous by Cooper (**4**) by the use of spherical Abelian means. Later, Bureau (**5**) found formulae equivalent to (4) for the odd and even cases separately by generalizing the definition of the Fourier transforms occurring in Courant's and Cooper's methods to include the finite and logarithmic parts of divergent integrals.

The classic formulae for the solution of the non-homogeneous wave-equation

$$L(u) = -g(x,t) \quad \text{(B)}$$

with homogeneous initial conditions are

(vi) the case $k = 1$,

$$u(x,t) = \tfrac{1}{2}c\iint g(\xi,\tau)\,d\xi d\tau, \quad (5)$$

where the integral is evaluated over the triangle bounded by the straight lines

$$c(t-\tau)\pm(x-\xi) = 0, \qquad \tau = 0;$$

(vii) in the case $k = 2$,

$$u(x,y,t) = \frac{c}{2\pi}\iiint \frac{g(\xi,\eta,\tau)}{\sqrt{\{c^2(t-\tau)^2-(x-\xi)^2-(y-\eta)^2\}}}\,d\xi d\eta d\tau, \tag{6}$$

where the integral is evaluated over the region bounded by the cone

$$c^2(t-\tau)^2-(x-\xi)^2-(y-\eta)^2 = 0$$

and the plane $\tau = 0$;

(viii) in the case $k = 3$, the retarded potential,

$$u(x,y,z,t) = \frac{1}{4\pi}\iiint_{r=ct} \frac{g(\xi,\eta,\zeta,t-r/c)}{r}\,d\xi d\eta d\zeta, \tag{7}$$

where $$r^2 = (x-\xi)^2+(y-\eta)^2+(z-\zeta)^2;$$

(ix) in the cases $k > 3$, taking $c = 1$,

$$u(x,t) = \frac{1}{(k-2)!}\left(\frac{\partial}{\partial t}\right)^{k-2}\int_0^t (t^2-r^2)^{\frac{1}{2}(k-3)} r M_r[g(r,t)]\,dr, \tag{8}$$

where $$r^2 = \sum_{i=1}^{k}(x_i-\xi_i)^2$$

(**3**, 402).

2. The Fourier transform method

Courant's method is essentially a method for calculating the elementary solution of the partial differential equation (A) and then using what Schwartz (**6** (ii), 67) has since shown to be a general property of convolutions of distributions. This property ensures that any partial differential equation can be solved once the elementary solution is known. Gates (**7**) has applied it to the case of the initial value problem for the equation of heat conduction in one dimension.

The elementary solution has already been given by Garnir (**9** and **8**, 278) for the cases $k = 1, 2, 3$, and has been given by Schwartz (cf. **6** (ii), 146) for all k in the form

$$-\frac{1}{4\pi^2}\overline{\mathscr{F}}\,Pf\frac{1}{\sigma^2}, \qquad \sigma^2 = c^2t^2-r^2,$$

where $\overline{\mathscr{F}}$ denotes the inverse Fourier transform in the sense of the theory of distributions with respect to both space and time coordinates, and $Pf\sigma^{-2}$ denotes the distribution corresponding to the operation of taking Hadamard's 'finite part' of the integrals of products of σ^{-2} and test-functions. Garnir finds his elementary solutions by a Laplace transform method somewhat similar to the Fourier transform method used in this paper. Gates

merely verifies that a certain distribution is an elementary solution. In Bureau's method for the classical case of functions the elementary solution is the same as that of Hadamard, but this is not necessarily the same as the elementary solution in distributions (cf. **6** (i), 133).

Following Courant's method, in this paper the elementary solutions are calculated directly by considering Fourier transforms with respect to space-coordinates only and in all cases the formulae

$$c\mathscr{F}\frac{\sin 2\pi c\rho t}{2\pi\rho}, \qquad \rho = \sum_{i=1}^{k} \xi_i^2$$

found, where $\mathscr{F}$ denotes the Fourier transform with respect to space-coordinates only. This expression is easier to evaluate than Schwartz's.

The case of distributions subsumes all the classic cases so far studied, and the advantage of dealing with them instead of with the classical functions is that all difficulties with regard to convergence and differentiability of integrals are eliminated. Thus the formal operations of Courant are immediately justified and the cases of odd and even numbers of variables become similar. As compared with the method of Bureau the method of distributions has the advantage that it leads directly to the final expressions in terms of convergent integrals without having to consider finite and logarithmic parts.

3. The theory of distributions

The theory of distributions (**6**) is essentially a theory of weak convergence and for our purpose may be summarized as follows:

A distribution is a linear functional on the space $(\mathscr{D})$ of infinitely differentiable functions of compact support. Test-functions ('fonctions indéfiniment dérivables à décroissance rapide' (**6** (ii), 89) are defined by the conditions that

(i) every test-function $\phi(x)$ is infinitely differentiable at all points x in k-dimensional space;

(ii) $$\lim_{|x|\to\infty} \left| x_1^l x_2^m \dots x_k^p \frac{\partial^{\alpha+\beta+\dots+\mu}}{\partial x_1^\alpha \partial x_2^\beta \dots \partial x_k^\mu} \phi(x) \right| = 0$$

for all positive integers $l, m, \dots, p, \alpha, \beta, \dots, \mu$.

A parameter function of slow growth with respect to α ('fonction tempérée' (**6** (ii), 93)), i.e. any function of the form

$$\frac{\partial^{p+q+\dots+t}}{\partial\alpha_1^p \partial\alpha_2^q \dots \partial\alpha_k^t}\{(1+|\alpha|^2)^{\frac{1}{2}k} f(x,\alpha)\}, \qquad |f(x,\alpha)| < K,$$

where all the indices are positive integers or zero and $k > 0$, is said to

converge to the limit $l(x)$ in the space of distributions of slow growth as α tends to α_1, if

$$\lim_{\alpha\to\alpha_1}\int [f(\alpha,x)-l(x)]\phi(x)\,dx = 0$$

in the Lebesgue sense for all test-functions $\phi(x)$, the integral being taken over the support of $\phi(x)$. In particular, the Fourier transform of a bounded function $f(x)$

$$\mathscr{F}f(x) = \int e^{2\pi i\mathbf{x}.\boldsymbol{\xi}}f(x)\,dx, \qquad \mathbf{x}.\boldsymbol{\xi} = \sum_{i=1}^{k} x_i\xi_i,$$

converges in the sense of the theory of distributions to the limit $F(\xi)$ if

$$\int [\mathscr{F}f(x)-F(\xi)]\phi(\xi)\,d\xi = 0$$

for all test-functions.

The principal properties of distributions used in this paper are the following:

(i) A distribution of slow growth has an infinite number of partial derivatives, where the operation of differentiation is defined for distributions as in (**6** (i), Ch. 2) and, in the cases where the distribution is a function differentiable in the ordinary sense, the derivative as defined for distributions is identical with the ordinary derivative. In particular, an integral containing a parameter may be differentiated with respect to that parameter if it can be shown to converge in the sense of the theory of distributions. This property will ensure the validity of all the operations performed on Fourier transforms in Courant's method.

(ii) The Dirac delta function

$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases} \qquad \int \delta(x)\,dx = 1,$$

and its derivatives are 'measures' and may be integrated and differentiated without restriction when associated with any continuous function of compact support (**6** (i), 11).

(iii) If the Fourier transform of a function converges in the sense of the theory of distributions, the original function is the unique inverse Fourier transform of that transform (**6** (ii), 107).

(iv) If L is the distribution corresponding to a given differential operator and the distribution E satisfies the differential equation $LE = \delta$, where δ is the Dirac measure in the space of all the variables concerned, the distribution $E * B$, defined in the case where E and B are integrable functions as

$$E * B = \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\xi_1 \int_{-\infty}^{\infty} d\xi_2 \dots \int_{-\infty}^{\infty} d\xi_k\, E(x-\xi,\ t-\tau)B(\xi,\tau)$$

and known as the convolution of E and B, satisfies the differential equation $LT = B$, where B is any given distribution (**6** (ii), 6 and 67). When B is not an integrable function the definition of $E * B$ can be suitably generalized but this will not be needed here.

(v) Garnir (**10**, 90) proves that where the solution of a partial differential equation in the sense of the theory of distributions satisfies the requirements for a solution in the ordinary sense it is identical with the solution in the ordinary sense.

4. The elementary solutions

An elementary solution of either of the differential equations (A) or (B) in the theory of distributions is any distribution E which satisfies the relation

$$LE = \delta,$$

where δ is the Dirac measure for the space of all the variables occurring in the differential equation.

Assume (following the method of Schwartz (**6** (ii), 142) but taking Fourier transforms with respect to space coordinates only) that

$$E(x,t) = \overline{\mathscr{F}}\,\mathscr{E}(\xi,t),$$

where $\overline{\mathscr{F}}$ denotes the inverse Fourier transform with respect to space coordinates. By differentiation under the integral sign we have

$$LE = \overline{\mathscr{F}}\left[-4\pi^2\rho^2\mathscr{E} - \frac{1}{c^2}\frac{\partial^2\mathscr{E}}{\partial t^2}\right] = \delta(x)\delta(t), \qquad \rho^2 = \sum_{i=1}^{k}\xi_i^2,$$

and, taking the Fourier transform of this equation, we have

$$\frac{\partial^2\mathscr{E}}{\partial t^2} + 4\pi^2c^2\rho^2\mathscr{E} = -c^2\delta(t).$$

This differential equation can be solved by the method of variation of parameters, giving the elementary solution

$$E(x,t) = \overline{\mathscr{F}}\,\mathscr{E}(\xi,t), \qquad \mathscr{E}(\xi,t) = -cH(t)\frac{\sin 2\pi c\rho t}{2\pi\rho},$$

where

$$H(t) \equiv \begin{cases} 0, & \text{if } t < 0, \\ 1, & \text{if } t \geqslant 0. \end{cases}$$

When $k = 1$,

$$E(x,t) = -cH(t)\mathscr{F}\frac{\sin 2\pi c\xi t}{2\pi\xi} = -cH(t)\int_{-\infty}^{\infty} e^{-2\pi i\xi x}\frac{\sin 2\pi c\xi t}{2\pi\xi}\,d\xi$$

$$= -\frac{c}{2}H(ct-|x|)$$

by contour integration.

When $k = 2$,

$$H(t)\mathscr{F}\mathscr{E}(\xi,t) = -cH(t)\int_0^{\infty}\rho\,d\rho\int_0^{2\pi} e^{-2\pi ir\rho\cos\theta}\frac{\sin 2\pi c\rho t}{2\pi\rho}\,d\theta$$

$$= -c^2H(t)\int_0^{\infty}\sin 2\pi c\rho t\,J_0(2\pi r\rho)\,d\rho$$

$$= -\frac{c}{2\pi}\frac{H(ct-r)}{\sqrt{(c^2t^2-r^2)}},$$

by Weber's formula (**11**, 405).

When $k = 3$,

$$H(t)\mathscr{F}\mathscr{E}(\xi,t) = -2\pi cH(t)\int_0^{\infty}\rho^2\,d\rho\int_0^{\pi} e^{-2\pi ir\rho\cos\theta}\frac{\sin 2\pi c\rho t}{2\pi\rho}\,d\theta$$

$$= -c\frac{H(t)}{2\pi\rho}\int_0^{\infty}\sin 2\pi c\rho t\sin 2\pi r\rho\,d\rho$$

$$= -c\frac{H(t)}{2\pi\rho}\int_0^{\infty}[\cos 2\pi(ct-r)\rho-\cos 2\pi(ct+r)\rho]\,d\rho.$$

The integral is divergent in the classical sense but may be calculated in distributions as follows. Consider the integral

$$\int_0^{R}\cos\alpha\rho\,d\rho = \left|\frac{\sin\alpha\rho}{\alpha}\right|_0^{R} = \frac{\sin\alpha R}{\alpha} \qquad (\alpha\neq 0),$$

and also the integral

$$I(R) = \int_{-\infty}^{\infty}\phi(\alpha)\sin\alpha R\,d\alpha,$$

where $\phi(\alpha)$ is any test-function. Now

$$I(R) = \left| -\frac{\phi(\alpha)\cos\alpha R}{R} \right|_{-\infty}^{\infty} + \frac{1}{R}\int_{-\infty}^{\infty} \phi'(\alpha)\cos\alpha R\, d\alpha$$

and therefore $\lim_{R\to\infty} I(R) = 0$, since

$$\left| \int_{-\infty}^{\infty} \phi'(\alpha)\sin\alpha R\, d\alpha \right| < \int_{-\infty}^{\infty} |\phi'(\alpha)|\, d\alpha < K,$$

so that, in the sense of the theory of distributions,

$$-H(t)\int_0^{\infty} [\cos 2\pi(ct-r)\rho - \cos 2\pi(ct+r)\rho]\, d\rho = \begin{cases} 0, & \text{if } ct-r \neq 0, \\ -\infty, & \text{if } ct-r = 0. \end{cases}$$

This suggests that

$$H(t)\overline{\mathscr{F}}\mathscr{E}(\xi,t) = -\frac{cK}{2\pi r}\delta(ct-r)$$

and K is easily found to be $\frac{1}{2}$ by considering the inverse relation

$$\mathscr{F}\frac{K}{2\pi r}\delta(ct-r) = c\frac{\sin 2\pi c\rho t}{2\pi\rho}, \quad t > 0.$$

When $k = 3$, therefore,

$$E(x,t) = -\frac{c}{4\pi}\frac{\delta(ct-r)}{r}$$

is an elementary solution.

When $k > 3$,

$$\overline{\mathscr{F}}\mathscr{E}(\xi,t) = -c\frac{(2\pi)^{\frac{1}{2}k}}{(2\pi r)^{\frac{1}{2}(k-2)}}\int_0^{\infty} \frac{\sin 2\pi c\rho t}{2\pi\rho}\rho^{\frac{1}{2}k} J_{\frac{1}{2}(k-2)}(2\pi r\rho)\, d\rho$$

$$= -\frac{c}{r^{\frac{1}{2}(k-2)}}\int_0^{\infty} \sin 2\pi c\rho t\, \rho^{\frac{1}{2}(k-2)} J_{\frac{1}{2}(k-2)}(2\pi r\rho)\, d\rho.$$

Again the integral is divergent in the ordinary sense, but may be shown to have a meaning in the theory of distributions as follows: we have

$$J_{\frac{1}{2}(k-2)}(2\pi r\rho) = \frac{2(\pi r\rho)^{\frac{1}{2}(k-2)}}{\Gamma[\frac{1}{2}(k-1)]\Gamma(\frac{1}{2})}\int_0^1 (1-u^2)^{\frac{1}{2}(k-3)}\cos 2\pi r\rho u\, du$$

by Lommel's formula (**11**, 48). Now, before proceeding to the limit in the formal expression

$$\bar{\mathscr{F}}\mathscr{E}(\xi,t)$$

$$= -\lim_{R\to\infty}\frac{c\pi^{\frac{1}{2}(k-2)}}{\Gamma[\frac{1}{2}(k-1)]\Gamma(\frac{1}{2})}\int\limits_0^R \rho^{k-2}\sin 2\pi c\rho t\, d\rho \int\limits_0^1 (1-u^2)^{\frac{1}{2}(k-3)}\cos 2\pi r\rho u\, du$$

consider instead of the double integral the equivalent double integral

$$I(R) = \int\limits_0^1 (1-u^2)^{\frac{1}{2}(k-3)}\, du \int\limits_0^R \rho^{k-2}\,\tfrac{1}{2}[\sin 2\pi(ct-ru)\rho+\sin 2\pi(ct+ru)\rho]\, d\rho.$$

The inner integral is of the form

$$\int\limits_0^R \rho^p \sin\alpha\rho\, d\rho = \left|-\rho^p\frac{\cos\alpha\rho}{\alpha}\right|_0^R+\frac{p}{\alpha}\int\limits_0^R \rho^{p-1}\cos\alpha\rho\, d\rho$$

$$= \left|-\rho^p\frac{\cos\alpha\rho}{\alpha}\right|_0^R+\frac{p}{\alpha}\left|\rho^{p-1}\frac{\sin\alpha\rho}{\alpha}\right|_0^R-$$

$$-\frac{(p-1)p}{\alpha^2}\int\limits_0^R \rho^{p-2}\sin\alpha\rho\, d\rho,$$

and so on. In the theory of distributions

$$\lim_{\rho\to\infty}\rho^p\sin\alpha\rho = 0 \quad\text{and}\quad \lim_{\rho\to\infty}\rho^p\cos\alpha\rho = 0, \quad p>0,\ \alpha\neq 0.$$

For, if $\phi(\alpha)$ is any test-function,

$$\int\limits_{-\infty}^{\infty}\phi(\alpha)\rho^p\sin\alpha\rho\, d\alpha = \rho^{p-1}\left|-\phi(\alpha)\cos\alpha\rho\right|_{-\infty}^{\infty}+\rho^{p-1}\int\limits_{-\infty}^{\infty}\phi'(\alpha)\cos\alpha\rho\, d\alpha$$

$$= \rho^{p-1}\int\limits_{-\infty}^{\infty}\phi'(\alpha)\cos\alpha\rho\, d\alpha$$

and

$$\int\limits_{-\infty}^{\infty}\phi(\alpha)\rho^p\cos\alpha\rho\, d\alpha = \rho^{p-1}\left|\phi(\alpha)\sin\alpha\rho\right|_{-\infty}^{\infty}-\rho^{p-1}\int\limits_{-\infty}^{\infty}\phi'(\alpha)\sin\alpha\rho\, d\alpha$$

$$= -\rho^{p-1}\int\limits_{-\infty}^{\infty}\phi'(\alpha)\sin\alpha\rho\, d\alpha.$$

A repeated application of this process leads to the inequalities

$$\left|\int\limits_{-\infty}^{\infty}\phi(\alpha)\rho^p\sin\alpha\rho\, d\alpha\right| \leqslant \rho^{-1}\int\limits_{-\infty}^{\infty}|\phi^{(p+1)}(\alpha)|\, d\alpha < K\rho^{-1}$$

and

$$\left|\int\limits_{-\infty}^{\infty}\phi(\alpha)\rho^p\cos\alpha\rho\, d\alpha\right| \leqslant \rho^{-1}\int\limits_{-\infty}^{\infty}|\phi^{(p+1)}(\alpha)|\, d\alpha < K\rho^{-1},$$

so that the limits as ρ tends to infinity are zero. Consequently, in the sense of the theory of distributions, by repeated integration by parts

$$\lim_{R\to\infty}\int_0^R \rho^p \sin\alpha\rho\, d\rho = \lim_{R\to\infty}(-1)^p\frac{p!}{\alpha^p}\int_0^R \begin{matrix}\sin\alpha\rho\\ \cos\alpha\rho\end{matrix}\, d\rho$$

$$= \begin{cases}0, & \text{if } \alpha \neq 0,\\ \infty & \text{with pole of order } p+1, \text{ if } \alpha = 0.\end{cases}$$

This suggests that

$$\lim_{R\to\infty}\int_0^R \rho^p \sin\alpha\rho\, d\rho = K\,\delta^{(p)}(\alpha),$$

where $\delta^{(p)}(\alpha)$ is the pth derivative of the Dirac measure; and

$$-\lim_{R\to\infty} H(t)I(R) = K\int_0^1 (1-u^2)^{\frac{1}{2}(k-3)}\delta^{(k-2)}(ct-ru)\frac{r\,du}{r}$$

$$= \begin{cases}\dfrac{K}{r}\left(\dfrac{\partial}{\partial t}\right)^{k-2}\left(1-\dfrac{c^2t^2}{r^2}\right)^{\frac{1}{2}(k-3)}, & \text{if } r < ct,\\ 0, \quad \text{if } r \geqslant ct,\end{cases}$$

or, choosing K to be real,

$$= \begin{cases}\dfrac{K}{r}\left(\dfrac{\partial}{\partial t}\right)^{k-2}\left(\dfrac{c^2t^2}{r^2}-1\right)^{\frac{1}{2}(k-3)}, & \text{if } r < ct\\ 0, \quad \text{if } r \geqslant ct.\end{cases}$$

The constant K is determined by the inverse relation

$$\mathscr{F}\frac{K}{r}\left(\frac{\partial}{\partial t}\right)^{k-2}\left(\frac{c^2t^2}{r^2}-1\right)^{\frac{1}{2}(k-3)} = -c\frac{\sin 2\pi c\rho t}{2\pi\rho} \quad (r > ct, t > 0),$$

and after a somewhat tedious calculation involving Lommel's function (cf. **11**, 374, 345) it is found to be

$$-\frac{1}{\omega_k c^{k-3}(k-2)!},$$

where ω_k is the area of the unit hypersphere. The elementary solution for $k > 3$ is thus

$$-\frac{1}{\omega_k c^{k-3}(k-2)!}\,\frac{1}{r}\left(\frac{\partial}{\partial t}\right)^{k-2}\left(\frac{c^2t^2}{r^2}-1\right)^{\frac{1}{2}(k-3)} H(ct-r).$$

5. The initial value problems

The solutions of the initial value problems may now be found by means of the convolution theorem (iv) of (**4**). In the initial value problems

$$B = [u_0(x)\,\delta'(t)+u_1(x)\,\delta(t)]/c^2,$$

as may be seen from the Gaussian integral theorem for any self-adjoint differential operator of the second order (cf. **3**, 435)

$$\int_{(V)} \cdots \int (\phi\, L[u] - u\, L[\phi])\, dx_1\, dx_2 \ldots dx_k\, d\tau$$
$$= \int_{(S)} \cdots \int \left(\phi \sum a_{ik} \frac{\partial u}{\partial x_k} - u \sum a_{ik} \frac{\partial \phi}{\partial x_k}\right) \frac{\partial x_i}{\partial \nu}\, da$$

in which the given values $u_0(x)$ of u are associated with the transversal derivatives of ϕ on S and the given values $u_1(x)$ of the transversal derivatives du/dt are associated with ϕ itself.

When $k = 1$,

$$E(x,t) = -\frac{c}{2} H(ct - |x|),$$

$$E(x,t) * \frac{u_0(x)\,\delta'(t)}{c^2} = -\frac{1}{2c} \int_{-\infty}^{\infty} \delta'(\tau)\, d\tau \int_{x-c(t-\tau)}^{x+c(t-\tau)} u_0(\xi)\, d\xi$$
$$= \tfrac{1}{2}[u_0(x+ct) + u_0(x-ct)]$$

and

$$E(x,t) * \frac{u_1(x)\,\delta(t)}{c^2} = -\frac{1}{2c} \int_{-\infty}^{\infty} \delta(\tau)\, d\tau \int_{x-c(t-\tau)}^{x+c(t-\tau)} u_1(\xi)\, d\xi$$
$$= \frac{1}{2c} \int_{x-ct}^{x+ct} u_1(\xi)\, d\xi,$$

so that we recover d'Alembert's solution (1).

When $k = 2$,

$$E(x,t) = -\frac{c}{2\pi} \frac{H(ct-r)}{\sqrt{(c^2t^2 - r^2)}},$$

$$E(x,t) * \frac{u_0(x)\,\delta'(t)}{c^2} = -\frac{1}{2\pi c} \int_{-\infty}^{\infty} \delta'(\tau)\, d\tau \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{H[c(t-\tau) - \rho]\, u_0(x-\xi)}{\sqrt{\{c^2(t-\tau)^2 - \rho^2\}}}\, d\xi d\eta,$$

$$= -\frac{1}{2\pi c} \int_{-\infty}^{\infty} \delta'(\tau)\, \mu_{c(t-\tau)}(u_0)\, d\tau = \frac{1}{2\pi c} \frac{d}{dt} \mu_{ct}(u_0)$$

where $\rho^2 = (x-\xi)^2 + (y-\eta)^2$, and

$$E(x,t) * \frac{u_1(x)\,\delta(t)}{c^2} = \frac{1}{2\pi c} \mu_{ct}(u_1),$$

so that we recover the Poisson–Parseval solution (3), section 1.

When $k = 3$,
$$E(x,t) = -\frac{c}{4\pi}\frac{\delta(ct-r)}{r},$$

$$E(x,t) * \frac{u_0(x)\,\delta'(t)}{c^2} = -\frac{1}{4\pi c}\int\limits_{-\infty}^{\infty}\delta'(\tau)\,d\tau \iiint\limits_{-\infty}^{\infty}\frac{\delta[c(t-\tau)-\rho]}{\rho}u_0(\xi)\rho^2\,d\rho d\Omega,$$

where $\rho^2 = (x-\xi)^2+(y-\eta)^2+(z-\zeta)^2$ and $d\Omega$ is the element of solid angle,

$$= -\frac{1}{4\pi c}\int\limits_{-\infty}^{\infty}\delta'(\tau)\,d\tau \iint\limits_{r=c|t-\tau|} c(t-\tau)u_0(\xi)\,d\Omega$$

$$= -\frac{1}{4\pi c}\int\limits_{-\infty}^{\infty}\delta'(\tau)\,4\pi c(t-\tau)M_{c|t-\tau|}(u_0)\,d\tau$$

$$= \frac{d}{dt}[tM_{ct}(u_0)],$$

and similarly
$$E(x,t) * \frac{u_1(x)\,\delta(t)}{c^2} = tM_{ct}(u_1).$$

When $k > 3$,
$$E(x,t) = -\frac{1}{\omega_k c^{k-3}(k-2)!}\frac{1}{r}\left(\frac{\partial}{\partial t}\right)^{k-2}\left(\frac{c^2t^2}{r}-1\right)^{\frac{1}{2}(k-3)}H(ct-r),$$

$$E(x,t) * \frac{u_0(x)\,\delta'(t)}{c^2} = \frac{1}{c^{k-1}(k-2)!}\left(\frac{\partial}{\partial t}\right)^{k-3}\int\limits_0^{ct}(c^2t^2-\rho^2)^{\frac{1}{2}(k-3)}\rho M_\rho(u_0)\,d\rho$$

after an easy calculation, and

$$E(x,t) * \frac{u_1(x)\,\delta(t)}{c^2} = \frac{1}{c^{k-1}(k-2)!}\left(\frac{\partial}{\partial t}\right)^{k-2}\int\limits_0^{ct}(c^2t^2-\rho^2)^{\frac{1}{2}(k-3)}\rho M_\rho(u_1)\,d\rho,$$

so that

$$u(x,t) = \frac{1}{c^{k-1}(k-2)!}\left\{\left(\frac{\partial}{\partial t}\right)^{k-3}\int\limits_0^{ct}(c^2t^2-r^2)^{\frac{1}{2}(k-3)}rM_r(u_0)\,dr + \right.$$
$$\left. +\left(\frac{\partial}{\partial t}\right)^{k-2}\int\limits_0^{ct}(c^2t^2-r^2)^{\frac{1}{2}(k-3)}rM_r(u_1)\,dr\right\},$$

a generalization of (4), section 1.

6. The inhomogeneous problem

The solution of the inhomogeneous problem is given by the formula

$$u(x,t) = -E * g$$

which leads immediately to the formulae (5), (6), and (7), and the following generalization of (8)

$$u(x,t) = \frac{1}{c^{k-1}(k-2)!}\left(\frac{\partial}{\partial t}\right)^{k-2} \int\limits_0^{ct} (c^2t^2-r^2)^{\frac{1}{2}(k-3)} r M_r\{g(r,t)\}\, dr.$$

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PROPAGATION IN A FERRITE-FILLED WAVEGUIDE

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SUMMARY

Perturbation methods are used for the solution of the fields in a waveguide filled with ferrite material which is subjected to a static magnetic field in the direction of its axis. It is shown that quasi TE and quasi TM modes exist and the first terms in the expansion are calculated for the case of a waveguide of rectangular cross-section.

1. Introduction

It has been found experimentally that, when a static magnetic field is impressed on materials such as ferrites, they become anisotropic. The mechanism by which this happens is not altogether known but a possible explanation has been given by Polder (**1**). The atoms comprising such substances are arranged in a cubic lattice and behave as magnetic tops, their angular momentum and magnetic moment being parallel. If an external field is applied the tops will align themselves in certain directions, and if small disturbances such as a microwave field are superimposed upon the static magnetic field, the additional magnetic induction and the additional magnetic field intensity due to this microwave field obey the following relation:

$$B_x = \mu H_x - j\mu' H_y, \qquad j^2 = -1$$

$$B_y = j\mu' H_x + \mu H_y,$$

$$B_z = \mu_z H_z. \tag{1.1}$$

A time factor $\exp\{i\omega t\}$ is assumed and the static magnetic field is in the z direction. The exact manner in which μ, μ', μ_z behave with this applied static magnetic field is not important from the point of view of this paper, and has been given by Polder (**1**) and Van Trier (**3**). It may be noted, however, that (μ'/μ) is proportional to this applied field.

If a waveguide filled with ferrite be subjected to a static magnetic field in the direction of its axis, it is possible by using Maxwell's equations to determine the transverse components of the electric and magnetic intensities in terms of the longitudinal components. Without going into details

of the analysis which is given elsewhere (**2**), the field is given by the equations

$$(\kappa^4-k'^4)\mathbf{E}_t = \nabla(\kappa^2\gamma E_z+\omega\gamma^2\mu' H_z)+$$
$$+j\,\mathbf{i}_z\times\nabla\{\omega(\kappa^2\mu-k'^2\mu')H_z-\gamma k'^2E_z\}, \tag{1.2}$$
$$(\kappa^4-k'^4)\mathbf{H}_t = \nabla(\kappa^2\gamma H_z+\omega k'^2\epsilon E_z)-$$
$$-j\,\mathbf{i}_z\times\nabla\{\kappa^2\omega\epsilon E_z+\gamma k'^2 H_z\}, \tag{1.3}$$

where a factor $\exp(\gamma z)$ is assumed, $\mathbf{i}_z$ is a unit vector along the z direction, E_z, H_z are the field components in the z direction, $\mathbf{E}_t$, $\mathbf{H}_t$ are the field components in the transverse directions, and

$$\kappa^2 = \omega^2\epsilon\mu+\gamma^2, \qquad k'^2 = \omega^2\epsilon\mu', \tag{1.4}$$

where ϵ is the dielectric constant, ∇ is the planar gradient vector. The values of E_z and H_z are determined by means of the two differential equations

$$\nabla^2E_z+\{\kappa^2-(k'^2\mu')/\mu\}E_z-\omega\gamma\mu_z(\mu'/\mu)H_z = 0, \tag{1.5a}$$
$$\nabla^2H_z+\kappa^2(\mu_z/\mu)H_z+\omega\gamma\epsilon(\mu'/\mu)E_z = 0, \tag{1.5b}$$

subject to the boundary conditions

$$* \qquad E_z = 0, \tag{1.6a}$$
$$* \qquad jk'^2\gamma\frac{\partial E_z}{\partial n}+\omega\gamma^2\mu'\frac{\partial H_z}{\partial s}-j\omega(\kappa^2\mu-k'^2\mu')\frac{\partial H_z}{\partial n} = 0, \tag{1.6b}$$

where $\partial/\partial n$ and $\partial/\partial s$ represent the normal and tangential derivatives on the boundary.

Because of the nature of the boundary condition (1.6 b) it has not yet proved possible to solve these equations generally. Solutions only appear to be possible for waveguide cross-sections, such as the circle, for which it is possible to express an arc distance along the boundary in terms of one coordinate only.

However, if the ratio (μ'/μ) is small, it is possible to use perturbation methods, and a solution may be found in terms of a power series in (μ'/μ). The first terms of the series will now be found.

2. Preliminary analysis

It will be observed that in equations (1.5) and (1.6) the following quantities may be expanded in terms of (μ'/μ), namely H_z, E_z, γ. Now γ occurs both as γ and as γ^2, and it will accordingly be convenient to recast the equations so that γ^2, rather than γ, is the unknown quantity. There-

* Equations involving the boundary conditions will all be denoted in this manner. All other equations are applicable throughout the medium.

fore, writing $\omega\mu H_z = Q$, $\gamma E_z = P$, $k^2 = \omega^2\mu\epsilon$, $\alpha = \mu'/\mu$, and $\zeta = \mu_z/\mu$, equations (1.5) become

$$\nabla^2 P + \{\gamma^2 + k^2(1-\alpha^2)\}P + \zeta\alpha\gamma^2 Q = 0, \tag{2.1 a}$$

$$\nabla^2 Q + \{\gamma^2 + k^2\}Q - k^2\alpha P = 0, \tag{2.1 b}$$

subject to the boundary conditions on C,

$$P = 0, \tag{2.2 a}$$

$$\alpha\gamma^2 \frac{\partial Q}{\partial s} + jk^2\alpha \frac{\partial P}{\partial n} - j\{k^2(1-\alpha^2)+\gamma^2\}\frac{\partial Q}{\partial n} = 0. \tag{2.2 b}$$

It will be seen that, in this form, γ occurs only in the form γ^2. Now if $\alpha = 0$, equations (1.2), (1.3) become

$$\kappa^2 \mathbf{E}_t = \nabla\gamma E_z + j\,\mathbf{i}_z \times \nabla(\omega\mu H_z), \tag{2.3}$$

$$\kappa^2 \mathbf{H}_t = \nabla\gamma H_z - j\,\mathbf{i}_z \times \nabla(\omega\epsilon E_z), \tag{2.4}$$

the cross term in equation (2.1) disappears, and the fields split up into those for which $E_z = 0$ (TE) and those for which $H_z = 0$ (TM). Moreover the modes which occur in the ferrite-filled guides may be expected to be of two series, one of which reduces to the TE type, and the other to the TM type when $\alpha = 0$. Further, to a good degree of approximation (**3**), $\zeta = (\mu_z/\mu) = 1$ and, as the interest lies in the behaviour with α, this value for ζ will be assumed (the extension to $\zeta \neq 1$ will be almost obvious), whence the equations become

$$\nabla^2 P + \{\gamma^2 + k^2(1-\alpha^2)\}P + \alpha\gamma^2 Q = 0, \tag{2.5 a}$$

$$\nabla^2 Q + (\gamma^2 + k^2)Q - \alpha k^2 P = 0. \tag{2.5 b}$$

Before proceeding to the perturbation, a further mathematical tool is required—the Green's functions $G^{(1)}$, $G^{(2)}$ which are defined as follows:

$$\nabla^2 G^{(1,2)} + (k^2 + \gamma_0^2)G^{(1,2)} = \delta(x-x')\delta(y-y'), \tag{2.6 a}$$

$$G^{(1)} = 0, \qquad \frac{\partial G^{(2)}}{\partial n} = 0 \quad \text{on } C. \tag{2.6 b}$$

When these are known, the solution of

$$\nabla^2\chi + (k^2 + \gamma_0^2)\chi = f(x,y), \tag{2.7}$$

subject to the appropriate boundary condition, is given by

$$\chi(x,y) = \int_S f(x',y')G(x,x';y,y')\,dx'dy', \tag{2.8}$$

where the integration is over the cross-section of the waveguide. The Green's functions $G^{(1)}$, $G^{(2)}$ are calculated in Appendix A for a waveguide of rectangular cross-section.

3. Calculation of perturbed field

The field, and propagation constant, are assumed to be expressible in the form

$$P = \sum_{n=0}^{\infty} \alpha^n P_n, \tag{3.1}$$

$$Q = \sum_{n=0}^{\infty} \alpha^n Q_n, \tag{3.2}$$

$$\gamma^2 = \sum_{n=0}^{\infty} \alpha^n \gamma_n^2, \tag{3.3}$$

where γ_n^2 is not necessarily positive. We now substitute for P, Q and γ^2 in equations (2.5) and (2.2) and obtain the following formulae:

Equation (2.5 a) becomes

$$\sum_{n=0}^{\infty} \alpha^n(\nabla^2 P_n + k^2 P_n) - \alpha^2 k^2 \sum_{n=0}^{\infty} \alpha^n P_n + \\ + \sum_{n=0}^{\infty} \alpha^n \gamma_n^2 \sum_{m=0}^{\infty} \alpha^m P_m + \alpha \sum_{n=0}^{\infty} \alpha^n \gamma_n^2 \sum_{m=0}^{\infty} \alpha^m Q_m = 0$$

or

$$\sum_{n=0}^{\infty} \alpha^n \Big[(\nabla^2 + k^2) P_n + \sum_{r=0}^{n} \gamma_r^2 P_{n-r}\Big] - \\ - k^2 \sum_{n=2}^{\infty} \alpha^n P_{n-2} + \sum_{n=1}^{\infty} \alpha^n \sum_{r=0}^{n-1} \gamma_r^2 Q_{n-1-r} = 0,$$

whence

$$\nabla^2 P_0 + (k^2 + \gamma_0^2) P_0 = 0, \tag{3.40}$$

$$\nabla^2 P_1 + (k^2 + \gamma_0^2) P_1 = -\gamma_1^2 P_0 - \gamma_0^2 Q_0, \tag{3.41}$$

$$\nabla^2 P_n + (k^2 + \gamma_0^2) P_n = -\gamma_n^2 P_n - \sum_{r=1}^{n-1} \gamma_r^2 P_{n-r} - k^2 P_{n-2} - \sum_{r=0}^{n-1} \gamma_r^2 Q_{n-1-r}. \tag{3.42}$$

Equation (2.5 b) becomes

$$\sum_{n=0}^{\infty} \alpha^n (\nabla^2 + k^2) Q_n + \sum_{n=0}^{\infty} \alpha^n \gamma_n^2 \sum_{m=0}^{\infty} \alpha^m Q_m - \alpha k^2 \sum_{n=0}^{\infty} \alpha^n P_n = 0$$

or

$$\sum_{n=0}^{\infty} \alpha^n \Big[(\nabla^2 + k^2) Q_n + \sum_{r=0}^{n} \gamma_r^2 Q_{n-r}\Big] - k^2 \sum_{n=1}^{\infty} \alpha^n P_{n-1} = 0,$$

whence

$$\nabla^2 Q_0 + (k^2 + \gamma_0^2) Q_0 = 0, \tag{3.50}$$

$$\nabla^2 Q_1 + (k^2 + \gamma_0^2) Q_1 = -\gamma_1^2 Q_0 + k^2 P_0, \tag{3.51}$$

$$\nabla^2 Q_n + (k^2 + \gamma_0^2) Q_n = -\gamma_n^2 Q_0 - \sum_{r=1}^{n-1} \gamma_r^2 Q_{n-r} - k^2 P_{n-1}. \tag{3.52}$$

Equation (2.2 a) becomes

*

$$\sum_{n=0}^{\infty} \alpha^n P_n = 0,$$

whence

$$P_n = 0. \tag{3.62}$$

Equation (2.2 b) becomes

$$\alpha \sum_{n=0}^{\infty} \alpha^n \gamma_n^2 \sum_{m=0}^{\infty} \alpha^m \frac{\partial Q_m}{\partial s} + jk^2 \alpha \sum_{n=0}^{\infty} \alpha^n \frac{\partial P_n}{\partial n} -$$

$$-jk^2 \sum_{n=0}^{\infty} \alpha^n \frac{\partial Q_n}{\partial n} - j \sum_{n=0}^{\infty} \alpha^n \gamma_n^2 \sum_{m=0}^{\infty} \alpha^m \frac{\partial Q_m}{\partial n} + jk^2 \alpha^2 \sum_{m=0}^{\infty} \alpha^n \frac{\partial Q_n}{\partial n} = 0,$$

or

$$\sum_{n=1}^{\infty} \alpha^n \sum_{r=0}^{n-1} \gamma_r^2 \frac{\partial Q_{n-1-r}}{\partial s} + jk^2 \sum_{n=1}^{\infty} \alpha^n \frac{\partial P_{n-1}}{\partial n} - j \sum_{n=0}^{\infty} \alpha^n \sum_{r=0}^{n} \gamma_r^2 \frac{\partial Q_{n-r}}{\partial n} -$$

$$-jk^2 \sum_{n=0}^{\infty} \alpha^n \frac{\partial Q_n}{\partial n} + jk^2 \sum_{n=2}^{\infty} \alpha^n \frac{\partial Q_{n-2}}{\partial n} = 0;$$

whence

$$-j(k^2+\gamma_0^2) \frac{\partial Q_0}{\partial n} = 0, \tag{3.70}$$

$$\gamma_0^2 \frac{\partial Q_0}{\partial s} + jk^2 \frac{\partial P_0}{\partial n} - j(k^2+\gamma_0^2) \frac{\partial Q_1}{\partial n} - j\gamma_1^2 \frac{\partial Q_0}{\partial n} = 0, \tag{3.71}$$

$$\sum_{r=0}^{n-1} \gamma_r^2 \frac{\partial Q_{n-1-r}}{\partial s} + jk^2 \frac{\partial P_{n-1}}{\partial n} - j \sum_{r=0}^{n} \gamma_r^2 \frac{\partial Q_{n-r}}{\partial n} - jk^2 \frac{\partial Q_n}{\partial n} + jk^2 \frac{\partial Q_{n-2}}{\partial n} = 0, \tag{3.72}$$

It will be observed that equations (3.42), (3.52), (3.62), (3.72) determine P_n, Q_n, γ_n^2 in terms of the boundary conditions and the lower P_n, Q_n, γ_n^2. Thus from P_0, Q_0 it will be possible to build up the quantities P_n, Q_n.

The terms of zero order are

$$\nabla^2 P_0 + (k^2+\gamma_0^2) P_0 = 0, \tag{3.40}$$

$$\nabla^2 Q_0 + (k^2+\gamma_0^2) Q_0 = 0, \tag{3.50}$$

$$P_0 = 0, \tag{3.60}$$

$$-j(k^2+\gamma_0^2) \frac{\partial Q_0}{\partial n} = 0, \tag{3.70}$$

which is equivalent to $$\frac{\partial Q_0}{\partial n} = 0.$$

These represent the TE and TM modes for a guide filled with a material whose electrical constants are ϵ, μ but with $\mu' = 0$. (These will, of course, be the same in their cross-sectional behaviour as those for an empty guide.) On examination of the equations (3.4)–(3.7) it will be seen that any field for a ferrite filled guide will be composed of those modes such that

$P_0 = 0$ and those modes such that $Q_0 = 0$. Thus the solution of equations (3.4)–(3.7) may be considered separately for $P_0 = 0$ and for $Q_0 = 0$. Modes such that $P_0 = 0$ will be termed quasi TE and those for which $Q_0 = 0$, termed quasi TM. The first type will be dealt with in section 4 and the second type in section 5. (Equations in these sections which have appeared previously have their previous number on the left.)

4. Quasi TE modes ($P_0 = 0$)

The governing equations in this case are

(3.50) $$[\nabla^2+(k^2+\gamma_0^2)]Q_0 = 0, \tag{4.1 a}$$

*(3.70) $$\frac{\partial Q_0}{\partial n} = 0. \tag{4.1 b}$$

Q_0 will be taken to be any particular mode of the empty guide given by equation (4.1).

The first-order terms become

(3.41) $$\nabla^2 P_1+(k^2+\gamma_0^2)P_1 = -\gamma_0^2 Q_0, \tag{4.2 a}$$

(3.51) $$\nabla^2 Q_1+(k^2+\gamma_0^2)Q_1 = -\gamma_1^2 Q_0, \tag{4.2 b}$$

*(3.61) $$P_1 = 0, \tag{4.2 c}$$

*(3.71) $$\gamma_0^2\frac{\partial Q_0}{\partial s}-j(k^2+\gamma_0^2)\frac{\partial Q_1}{\partial n} = 0. \tag{4.2 d}$$

From equations (4.2 a) and (4.2 c) it is possible, using the Green's function $G^{(1)}$ to calculate P_1. This is done for a rectangular guide in Appendix B 1. There remain to be calculated γ_1^2 and Q_1. For the calculation of γ_1^2 the method of Caprioli may be used (**4**). From equations (4.1 a) and (4.2 b)

$$\int\limits_S (Q_0\nabla^2 Q_1 - Q_1\nabla^2 Q_0)\,dS = -\gamma_1^2\int\limits_S Q_0^2\,dS,$$

and by Green's theorem

$$\int\limits_S (Q_0\nabla^2 Q_1 - Q_1\nabla^2 Q_0)\,dS = \int\limits_C \left(Q_0\frac{\partial Q_1}{\partial n} - Q_1\frac{\partial Q_0}{\partial n}\right)ds = \int\limits_C Q_0\frac{\partial Q_1}{\partial n}\,ds$$

$$= -\frac{j\gamma_0^2}{(k^2+\gamma_0^2)}\int\limits_C Q_0\frac{\partial Q_0}{\partial s}\,ds = 0,$$

S being the cross-section of the guide and C of the wall. This follows because $Q_0\dfrac{\partial Q_0}{\partial s} = \dfrac{\partial}{\partial s}(\tfrac{1}{2}Q_0^2)$ and we integrate completely round C. Thus

$$\gamma_1^2\int Q_0^2\,dS = 0,$$

and so
$$\gamma_1^2 = 0. \tag{4.3}$$

Equation (4.2 b) becomes

$$\nabla^2 Q_1 + (k^2+\gamma_0^2) Q_1 = 0. \tag{4.4}$$

We have by Green's theorem

$$\int\limits_S [G^{(2)}\nabla^2 Q_1 - Q_1 \nabla^2 G^{(2)}]\, dS' = \int\limits_C \left[G^{(2)}\frac{\partial Q_1}{\partial n'} - Q_1 \frac{\partial G^{(2)}}{n'}\right] ds' \tag{4.5}$$

where S is the cross-section of the waveguide and C is the wall. Using equations (4.4), (2.6 a), (2.6 b) it follows that

$$-\int\limits_S Q_1(x',y')\delta(x-x')\,\delta(y-y')\, dx'dy' = \int G^{(2)}\frac{\partial Q_1}{\partial n'}\, ds',$$

whence

$$Q_1(x,y) = -\int\limits_C G^{(2)}\frac{\partial Q_1}{\partial n'}\, ds'$$

$$= \frac{j\gamma_0^2}{\gamma_0^2+k^2}\int\limits_C G^{(2)}(x,x';y,y')\,\frac{\partial Q_0(x',y')}{\partial s'}\, ds'. \tag{4.6}$$

This procedure is carried out for a rectangular guide in Appendix B 2.

Thus P_1, Q_1, γ_1^2 have all been determined and it is possible to go on to the terms of higher order.

Now equations (3.42), (3.52), (3.62), (3.72) may be regarded as expressing the P_n, Q_n, γ_n^2 in terms of the quantities of lower order. Remembering that $P_0 = 0$, equation (3.42) may be expressed as

$$[\nabla^2 + (k^2+\gamma_0^2)]P_n = F_{n-1}, \tag{4.7}$$

subject to

*(3.62) $$P_n = 0,$$

F_{n-1} being terms of order not greater than $n-1$, and the solution follows immediately by using the first Green's function as was done for P_1. The evaluation of γ_n^2 and Q_n presents a little more difficulty.

Using Caprioli's method again,

$$\int\limits_C Q_0\frac{\partial Q_n}{\partial n}\, ds = \int\limits_C \left(Q_0\frac{\partial Q_n}{\partial n} - Q_n\frac{\partial Q_0}{\partial n}\right) ds = \int\limits_S (Q_0\nabla^2 Q_n - Q_n\nabla^2 Q_0)\, dS,$$

which may be rewritten, using (3.52) and (3.72) as

$$\frac{1}{j(k^2+\gamma_0^2)}\int\limits_C Q_0\Bigg[\sum_{r=0}^{n-1}\gamma_r^2\frac{\partial Q_{n-1-r}}{\partial s}+jk^2\frac{\partial P_{n-1}}{\partial n}-$$
$$-j\sum_{r=0}^{n-1}\gamma_r^2\frac{\partial Q_{n-r}}{\partial n}+jk^2\frac{\partial Q_{n-2}}{\partial n}\Bigg]\,ds$$
$$= -\int\limits_S\Big[\gamma_n^2\,Q_0^2+k^2P_{n-1}Q_0+Q_0\sum_{r=0}^{n-1}\gamma_r^2\,Q_{n-r}\Big]\,dS, \tag{4.8}$$

whence γ_n^2 may be calculated.

To obtain Q_n write equation (3.72) in the form

$$* \quad \sum_{n=0}^{r-1}\gamma_r^2\frac{\partial Q_{n-1-r}}{\partial s}+jk^2\frac{\partial P_{n-1}}{\partial n}-j\sum_{r=1}^{n}\gamma_r^2\frac{\partial Q_{n-r}}{\partial n}+jk^2\frac{\partial Q_{n-2}}{\partial n}=j(k^2+\gamma_0^2)\frac{\partial Q_n}{\partial n}. \tag{4.9}$$

The left-hand side is known completely and so it follows in exactly the same way as equation (4.6) was obtained that

$$Q_n(x,y) = -\int\limits_C G^{(2)}\frac{\partial Q_n}{\partial n'}\,ds'. \tag{4.10}$$

5. Quasi TM modes ($Q_0 = 0$)

The fields in the quasi TM modes may be calculated in exactly the same way as those for quasi TE modes. It is, however, of interest to consider the first-order terms

(3.41) $$\nabla^2P_1+(k^2+\gamma_0^2)P_1 = -\gamma_1^2\,P_0, \tag{5.1 a}$$

*(3.61) $$P_1 = 0, \tag{5.1 b}$$

(3.51) $$\nabla^2Q_1+(k^2+\gamma_0^2)Q_1 = k^2P_0, \tag{5.1 c}$$

*(3.71) $$jk^2\frac{\partial P_0}{\partial n}-j(k^2+\gamma_0^2)\frac{\partial Q_1}{\partial n} = 0. \tag{5.1 d}$$

Applying Caprioli's method,

$$0 = \int\limits_C\left(P_1\frac{\partial P_0}{\partial n}-P_0\frac{\partial P_1}{\partial n}\right)ds = \int\limits_S (P_1\,\nabla^2P_0-P_0\,\nabla^2P_1)\,dS$$
$$= \gamma_1^2\int\limits_S P_0^2\,dS, \tag{5.2}$$

whence $$\gamma_1^2 = 0.$$

It follows from equations (3.41) and (2.8) that $P_1 = 0$.

In exactly the same way as was indicated previously in section 4

$$Q_1(x,y) = -\int G^{(2)} \frac{\partial Q_1}{\partial n'}\, ds'$$

$$= -\frac{\gamma_0^2}{k^2+\gamma_0^2} \int_C G^{(2)} \frac{\partial P_0}{\partial n'}\, ds'. \tag{5.3}$$

Calculation of this quantity for a waveguide of rectangular cross-section is given in Appendix B 3.

The higher order terms (i.e. $n \geqslant 2$) may be calculated in the manner indicated in section 4.

6. Discussion

The transverse fields may be calculated from equations (1.2) and (1.3), so the field is completely known. It will be observed that the terms of order α in the expansion of γ^2 vanish and thus, because $\alpha = (\mu'/\mu)$ and μ' is proportional to the applied magnetic field (**3**), the propagation constants in a waveguide filled with ferrites are insensitive to the effect of a small magnetic field.

The utility of this lies in the fact that once the quantities P_n, Q_n, γ_n^2 have been calculated, it is possible to find the fields for any arbitrary value of α. The quantities P_1 and Q_1 for the quasi TE case ($P_0 = 0$) and the quantity Q_1 for the quasi TM case ($Q_0 = 0$) have been calculated in the appendices for the case of a waveguide of rectangular cross-section. For this, an exact solution is difficult to obtain as a boundary condition of the form

$$A\frac{\partial Q}{\partial s} + B\frac{\partial P}{\partial n} + C\frac{\partial Q}{\partial n} = 0,$$

causes trouble, the difficulty arising from the fact that over part of the boundary n is in the x direction and over part in the y direction, and similarly for s. This difficulty does not arise in the case of a circular boundary where s is always in the θ direction and n is always in the r direction.

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APPENDIX A

Calculation of Green's functions for a rectangular guide

Let us suppose that the guide is given by $0 < x < a$, $0 < y < b$.

Let
$$\kappa_{rs}^2 = k^2+\gamma_0^2-\frac{r^2\pi^2}{a^2}-\frac{s^2\pi^2}{b^2}, \tag{A 1}$$

$$\begin{aligned}\epsilon_m &= 1 \quad m = 0,\\ &= 2 \quad m \neq 0.\end{aligned} \tag{A 2}$$

The Green's function of the second kind is given by
$$\frac{\partial^2 G^{(2)}}{\partial x^2}+\frac{\partial^2 G^{(2)}}{\partial y^2}+\kappa_{00}^2\, G^{(2)} = \delta(x-x')\delta(y-y') \tag{A 3}$$
and

* $$\frac{\partial G^{(2)}}{\partial n} = 0 \quad \text{over } x = 0,\ x = a;\ y = 0,\ y = b. \tag{A 4}$$

It follows from the boundary condition that $G^{(2)}$ must be a double Fourier series of cosines. Also
$$G^{(2)}(x,x';y,y') = G^{(2)}(x',x;y',y). \tag{A 5}$$
Accordingly we write
$$G^{(2)}(x,x';y,y') = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} G_{rs}\epsilon_r\epsilon_s \cos\left(\frac{r\pi x}{a}\right)\cos\left(\frac{r\pi x'}{a}\right)\cos\left(\frac{s\pi y}{b}\right)\cos\left(\frac{s\pi y'}{b}\right), \tag{A 6}$$
which satisfies all these conditions. Also
$$\frac{\partial^2 G^{(2)}}{\partial x^2}+\frac{\partial^2 G^{(2)}}{\partial y^2}+\kappa_{00}^2\, G^{(2)} = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \epsilon_r\,\epsilon_s\, G_{rs}\,\kappa_{rs}^2 \cos\left(\frac{r\pi x}{a}\right)\cos\left(\frac{r\pi x'}{a}\right)\cos\left(\frac{s\pi y}{b}\right)\cos\left(\frac{s\pi y'}{b}\right). \tag{A 7}$$

Multiplying both sides of the equation (A 3) by $\frac{1}{ab}\cos\left(\frac{m\pi x'}{a}\right)\cos\left(\frac{n\pi y'}{b}\right)$, integrating over $0 < x' < a$, $0 < y' < b$, and using the result
$$\frac{1}{a}\int_0^a \epsilon_r \cos\frac{r\pi x'}{a}\cos\frac{m\pi x'}{a}\,dx' = \delta_{rm}, \tag{A 8}$$
it follows that
$$ab\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} G_{rs}\kappa_{rs}^2\delta_{rm}\delta_{sn}\cos\left(\frac{r\pi x}{a}\right)\cos\left(\frac{s\pi y}{b}\right) = \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right), \tag{A 9}$$
whence
$$G_{mn} = (ab\kappa_{rs}^2)^{-1} \tag{A 10}$$
and
$$G^{(2)}(x,x';y,y') = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \epsilon_r\,\epsilon_s(ab\kappa_{rs}^2)^{-1}\cos\left(\frac{r\pi x}{a}\right)\cos\left(\frac{r\pi x'}{a}\right)\cos\left(\frac{s\pi y}{b}\right)\cos\left(\frac{s\pi y'}{b}\right). \tag{A 11}$$

In exactly the same way it may be proved that
$$G^{(1)}(x,x';y,y') = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} 4(ab\kappa_{rs}^2)^{-1}\sin\left(\frac{r\pi x}{a}\right)\sin\left(\frac{r\pi x'}{a}\right)\sin\left(\frac{s\pi y}{b}\right)\sin\left(\frac{s\pi y'}{b}\right). \tag{A 12}$$

This clearly vanishes on the boundary.

APPENDIX B

1. Calculation of P_1 for quasi TE mode ($P_0 = 0$)

The equations governing P_1 are

(4.2 a) $$\nabla^2 P_1 + \kappa_{00}^2 P_1 = -\gamma_0^2 Q_0, \tag{B 1.1}$$

(4.2 c)* $$P_1 = 0 \quad \text{on the boundary.} \tag{B 1.2}$$

The value of P_1 is given by

$$P_1(x,y) = \int_0^a \int_0^b G^{(1)}(x,x';y,y')\{-\gamma_0^2 Q_0(x',y')\}\,dx'dy'. \tag{B 1.3}$$

Now a typical TE mode is that given by

$$Q_0 = \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right), \tag{B 1.4}$$

so P_1 is given by

$$P_1 = -\gamma_0^2 \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} 4(ab\kappa_{rs}^2)^{-1} \times$$

$$\times \int_0^a \int_0^b \sin\left(\frac{r\pi x}{a}\right)\sin\left(\frac{r\pi x'}{a}\right)\sin\left(\frac{s\pi y}{b}\right)\sin\left(\frac{s\pi y'}{b}\right)\cos\left(\frac{m\pi x'}{a}\right)\cos\left(\frac{n\pi y'}{b}\right)dx'dy'$$

$$= -\gamma_0^2 \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} (\kappa_{rs}^2)^{-1} I_{rm} I_{sn} \sin\left(\frac{r\pi x}{a}\right)\sin\left(\frac{s\pi y}{b}\right), \tag{B 1.5}$$

where
$$I_{rm} = 2\int_0^1 \cos m\pi\zeta \sin r\pi\zeta \, d\zeta. \tag{B 1.6}$$

2. Calculation of Q_1 for quasi TE mode ($P_0 = 0$)

We have

(4.6) $$Q_1(x,y) = \frac{j\gamma_0^2}{\kappa_{00}^2} \int_C G^{(2)}(x,x';y,y') \frac{\partial Q_0(x',y')}{\partial s'}\,ds'. \tag{B 2.1}$$

This becomes, for the square cross-section,

$$\frac{\kappa_{00}^2}{j\gamma_0^2} Q_1(x,y) = \int_0^a G^{(2)}(x,x';y,0)\frac{\partial Q_0(x',0)}{\partial x'}\,dx' + \int_0^b G^{(2)}(x,a;y,y')\frac{\partial Q_0(a,y')}{\partial y'}\,dy' -$$

$$- \int_a^0 G^{(2)}(x,x';y,b)\frac{\partial Q_0(x',b)}{\partial x'}\,dx' - \int_b^0 G^{(2)}(x,0;y,y')\frac{\partial Q_0(0,y')}{\partial y'}\,dy',$$

each of the four integrals representing integration in the anticlockwise sense along one of the four walls. This may be rewritten in the form

$$\frac{\kappa_{00}^2}{j\gamma_0^2} Q_1 = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \epsilon_r \epsilon_s (ab\kappa_{rs}^2)^{-1} \cos\left(\frac{r\pi x}{a}\right)\cos\left(\frac{s\pi y}{b}\right) E_{rs}, \tag{B 2.2}$$

where

$$E_{rs} = \int\limits_0^a \cos\left(\frac{r\pi x'}{a}\right)\left[\frac{\partial Q_0(x',0)}{\partial x'} + \cos s\pi \frac{\partial Q_0(x',b)}{\partial x'}\right] dx' +$$

$$+ \int\limits_0^b \cos\left(\frac{s\pi y'}{b}\right)\left[\frac{\partial Q_0(0,y')}{\partial y'} + \cos r\pi \frac{\partial Q_0(a,y')}{\partial y'}\right] dx'.$$

Again using the fact that a typical value for Q_0 is

$$\cos\left(\frac{m\pi y}{a}\right)\cos\left(\frac{n\pi y}{b}\right),$$

$$E_{rs} = \int\limits_0^a \cos\left(\frac{r\pi x'}{a}\right)[1+\cos s\pi \cos n\pi]\left[-\frac{m\pi}{a}\right]\sin\left(\frac{m\pi x'}{a}\right) dx' +$$

$$+ \int\limits_0^b \cos\left(\frac{s\pi y'}{b}\right)[1+\cos r\pi \cos m\pi]\left[-\frac{n\pi}{b}\right]\sin\left(\frac{n\pi y'}{b}\right) dy'$$

$$= -\frac{m\pi}{2}[1+\cos s\pi \cos n\pi]I_{rm} - \frac{n\pi}{2}[1+\cos r\pi \cos m\pi]I_{sn},$$

whence

$$\frac{j\kappa_{00}^2 Q_1}{\gamma_0^2} = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \epsilon_r \epsilon_s (ab\kappa_{rs}^2)^{-1} \cos\left(\frac{r\pi x}{a}\right)\cos\left(\frac{s\pi y}{b}\right) \times$$

$$\times \left\{\frac{m\pi}{2}[1+\cos s\pi \cos n\pi]I_{rm} + \frac{n\pi}{2}[1+\cos r\pi \cos m\pi]I_{sn}\right\}.$$

3. Calculation of Q_1 for quasi TM mode ($Q_0 = 0$)

$$Q_1(x,y) = -\frac{\gamma_0^2}{\kappa_{00}^2} \int\limits_C G^{(2)}(x,x';y,y') \frac{\partial P_0(x',y')}{\partial n'} ds', \qquad \text{(B 3.1)}$$

whence

$$-\frac{\kappa_{00}^2}{\gamma_0^2} Q_1(x,y) = -\int\limits_0^a G^{(2)}(x,x';y,0) \frac{\partial P_0(x',0)}{\partial y'} dx' +$$

$$+ \int\limits_0^b G^{(2)}(x,a;y,y') \frac{\partial P_0(a,y')}{\partial x'} dy' +$$

$$+ \int\limits_a^0 G^{(2)}(x,x';y,b) \frac{\partial P_0(x',b)}{\partial y'} dx' - \int\limits_b^0 G^{(2)}(x,0;y,y') \frac{\partial P_0(0,y')}{\partial x'} dy',$$

each of the four integrals representing integration in the anti-clockwise sense along one of the four walls.

This may be rewritten in the following form:

$$\frac{\kappa_{00}^2}{\gamma_0^2} Q_1(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \epsilon_r \epsilon_s (ab\kappa_{rs}^2)^{-1} \cos\left(\frac{r\pi x}{a}\right)\cos\left(\frac{s\pi y}{b}\right) F_{rs}, \qquad \text{(B 3.2)}$$

where

$$F_{rs} = \int\limits_0^a \cos\left(\frac{r\pi x'}{a}\right)\left[\frac{\partial P_0(x',0)}{\partial y'} + \cos s\pi\,\frac{\partial P_0(x',b')}{\partial y'}\right] dx' +$$

$$+ \int\limits_0^b \cos\left(\frac{s\pi y'}{b}\right)\left[\frac{\partial P_0(0,y')}{\partial x'} + \cos r\pi\,\frac{\partial P_0(a,y')}{\partial x'}\right] dy'.$$

Again using the fact that a typical value for P_0 is

$$\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right),$$

$$F_{rs} = \int\limits_0^a \cos\left(\frac{r\pi x'}{a}\right)\sin\left(\frac{m\pi x'}{a}\right)\frac{n\pi}{b}[1+\cos s\pi\cos n\pi]\,dx' +$$

$$+ \int\limits_0^b \cos\left(\frac{s\pi y'}{b}\right)\sin\left(\frac{n\pi y'}{b}\right)\frac{m\pi}{a}[1+\cos r\pi\cos m\pi]\,dy',$$

$$= \frac{n\pi}{2}\frac{a}{b}I_{rm}[1+\cos s\pi\cos n\pi] + \frac{m\pi}{2}\frac{b}{a}I_{sn}[1+\cos r\pi\cos m\pi],$$

whence

$$\frac{\kappa_{00}^2}{\gamma_0^2}Q_1 = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\epsilon_r\epsilon_s(ab\kappa_{rs}^2)^{-1}\cos\left(\frac{r\pi x}{a}\right)\cos\left(\frac{s\pi y}{b}\right)\times$$

$$\times\left\{\frac{n\pi}{2}\frac{a}{b}I_{rm}[1+\cos s\pi\cos n\pi] + \frac{m\pi}{2}\frac{b}{a}I_{sn}[1+\cos r\pi\cos m\pi]\right\}.$$

TWO PROPERTIES OF SPHERICAL HARMONICS

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SUMMARY

The integral of the square of the gradient of a solid harmonic over a sphere is evaluated; the corresponding integral for the second derivatives is also evaluated, and the results are applied to an integral that includes the elastic energy in a strained sphere and the rate of dissipation in a viscous sphere.

A natural definition of the irregularity of a function over a sphere leads to the conclusion that the irregularity is stationary for small variations of the function when the function is a surface harmonic and that the irregularity of any function is greater than that of the lowest term in its expansion in surface harmonics.

THE first of the properties in question has been used in geophysics (**1**, **2**), but may have other applications. The first application given here is used in a later paper.

1. Let K_n be a solid harmonic of positive integral degree n, and put

$$K_n = r^n S_n. \tag{1}$$

We show that over a sphere $r = a$

$$\iint \left(\frac{\partial K_n}{\partial x_i}\right)^2 dS = \frac{n(2n+1)}{a^2} \iint K_n^2\, dS. \tag{2}$$

We have by Green's theorem

$$\iiint \left(\frac{\partial K_n}{\partial x_i}\right)^2 d\tau = \iint \frac{x_i}{r} K_n \frac{\partial K_n}{\partial x_i}\, dS = \frac{n}{a} \iint K_n^2\, dS \tag{3}$$

and also

$$= \iiint \frac{r^{2n}}{a^{2n-2}} \left(\frac{\partial K_n}{\partial x_i}\right)^2_{r=a} dr d\omega$$

$$= \frac{a^3}{2n+1} \iint \left(\frac{\partial K_n}{\partial x_i}\right)^2 d\omega, \tag{4}$$

whence the result follows, since $dS = a^2\, d\omega$.

Since $\partial K_n/\partial x_i$ is a solid harmonic of degree $n-1$, repetition gives the double summation (**1**, p. 74)

$$\iint \left(\frac{\partial^2 K_n}{\partial x_i\, \partial x_k}\right)^2 dS = \frac{(n-1)n(2n-1)(2n+1)}{a^4} \iint K_n^2\, dS. \tag{5}$$

1.1. Let u_{ik} be a tensor of the form (**1**, p. 81)

$$u_{ik} = F_1 \frac{\partial^2 K_n}{\partial x_i \partial x_k} + \tfrac{1}{2}F_2\left(x_i \frac{\partial K_n}{\partial x_k} + x_k \frac{\partial K_n}{\partial x_i}\right) + F_3 x_i x_k K_n + F_4 \delta_{ik} K_n, \tag{6}$$

where F_1, F_2, F_3, F_4 are functions of r. Then

$$u_{ii} = (nF_2 + r^2F_3 + 3F_4)K_n. \tag{7}$$

The deviatoric part

$$\begin{aligned} u'_{ik} &= u_{ik} - \tfrac{1}{3}u_{mm}\delta_{ik} \\ &= F_1 \frac{\partial^2 K_n}{\partial x_i \partial x_k} + \tfrac{1}{2}F_2\left(x_i \frac{\partial K_n}{\partial x_k} + x_k \frac{\partial K_n}{\partial x_i}\right) + F_3 x_i x_k K_n - \tfrac{1}{3}(nF_2 + r^2F_3)\delta_{ik} K_n. \end{aligned} \tag{8}$$

The scalar

$$\begin{aligned} u'_{ik}u'_{ik} &= F_1^2\left(\frac{\partial^2 K_n}{\partial x_i \partial x_k}\right)^2 + \left\{\frac{1}{2}\left(F_2 + \frac{2(n-1)F_1}{r}\right)^2 - \frac{2(n-1)^2}{r^2}F_1^2\right\}\left(r\frac{\partial K_n}{\partial x_i}\right)^2 + \\ &\quad + \{\tfrac{1}{6}n^2F_2^2 + \tfrac{4}{3}nr^2F_2F_3 + 2n(n-1)F_1F_3 + \tfrac{2}{3}r^2F_3^2\}K_n^2 \\ &= G_1^2\left\{\left(r^2\frac{\partial^2 K_n}{\partial x_i \partial x_k}\right)^2 - 2(n-1)^2\left(r\frac{\partial K_n}{\partial x_i}\right)^2 + \tfrac{1}{2}n^2(n-1)^2K_n^2\right\} + \\ &\quad + \tfrac{1}{2}G_2^2\left\{\left(r\frac{\partial K_n}{\partial x_i}\right)^2 - n^2K_n^2\right\} + \tfrac{2}{3}G_3^2 K_n^2, \end{aligned} \tag{9}$$

where

$$G_1 = \frac{F_1}{r^2}; \qquad G_2 = F_2 + 2(n-1)\frac{F_1}{r^2}; \qquad G_3 = \tfrac{3}{2}n(n-1)\frac{F_1}{r^2} + nF_2 + r^2F_3. \tag{10}$$

Then the integral over a sphere of radius r is

$$\iint u'_{ik}u'_{ik}\,dS = \{\tfrac{1}{2}n(n-1)(n+1)(n+2)G_1^2 + \tfrac{1}{2}n(n+1)G_2^2 + \tfrac{2}{3}G_3^2\} \iint K_n^2\,dS. \tag{11}$$

If u_{ik} is a stress, and thus $u'_{ik}u'_{ik}$ the Mises function, the G_1 terms represent the contribution from the hoop stresses, G_2 from the shear stress over horizontal planes, and G_3 from the difference between the radial stress and the mean of the greatest and least hoop stresses.

If u_i is a displacement, and

$$u_i = F\frac{\partial K_n}{\partial x_i} + Gx_i K_n, \tag{12}$$

the strain component e_{ik} is

$$e_{ik} = F\frac{\partial^2 K_n}{\partial x_i \partial x_k} + \frac{1}{2}\left(\frac{F'}{r} + G\right)\left(x_i\frac{\partial K_n}{\partial x_k} + x_k\frac{\partial K_n}{\partial x_i}\right) + x_i x_k \frac{G'}{r}K_n + G\delta_{ik}K_n \tag{13}$$

and we find, with $u_{ik} = e_{ik}$

$$G_1 = \frac{F}{r^2}; \qquad G_2 = \frac{F'}{r} + G + 2(n-1)\frac{F}{r^2};$$

$$G_3 = rG' + n\left(\frac{F'}{r} + G\right) + \frac{3}{2}\frac{n(n-1)}{r^2}F_1. \tag{14}$$

This way of expressing the strain components is much more convenient than those that introduce derivatives of K_n/r^{2n+1}. It seems to have been introduced by Love (**3**, p. 15). Another type of solution arises when there is tangential stress over the outer boundary (**2**).

2. In the same conditions, if θ, χ are polar coordinates, and f is a function of θ, χ, a natural estimate summarizing the general magnitude of f is $\iint f^2\,d\omega$. A natural estimate summarizing the rate of variation of f with position on the sphere is

$$I = \iint \left\{\left(\frac{\partial f}{\partial \theta}\right)^2 + \left(\frac{1}{\sin\theta}\frac{\partial f}{\partial \chi}\right)^2\right\} d\omega.$$

Then the ratio of these two expressions is a precisely defined quantity that can be regarded as a measure of the irregularity of f. It is unaltered by change of axes. We have for any function of position

$$\left(\frac{\partial \phi}{\partial \theta}\right)^2 + \left(\frac{1}{\sin\theta}\frac{\partial \phi}{\partial \chi}\right)^2 = a^2\left\{\left(\frac{\partial \phi}{\partial x_i}\right)^2 - \left(\frac{\partial \phi}{\partial r}\right)^2\right\}, \tag{1}$$

and if

$$f = S_n, \qquad \phi = \frac{K_n}{a^n} = r^n\frac{S_n}{a^n}, \tag{2}$$

$$I = a^{-2n}\iint \left\{\left(\frac{\partial K_n}{\partial x_i}\right)^2 - \left(\frac{nK_n}{a}\right)^2\right\} dS$$

$$= \frac{n(n+1)}{a^{2n}}\iint K_n^2\,d\omega = n(n+1)\iint f^2\,d\omega. \tag{3}$$

Further, by the methods of the calculus of variations, we find that if f is replaced by $f+\delta f$,

$$\tfrac{1}{2}\delta \iint \left\{\left(\frac{\partial f}{\partial \theta}\right)^2 + \left(\frac{1}{\sin\theta}\frac{\partial f}{\partial \chi}\right)^2 - \lambda f^2\right\}\sin\theta\,d\theta d\chi$$

$$= \int_0^{2\pi}\left[\sin\theta\frac{\partial f}{\partial \theta}\delta f\right]_{\theta=0}^{\pi} d\chi + \int_0^{\pi}\left[\frac{1}{\sin\theta}\frac{\partial f}{\partial \chi}\delta f\right]_{\chi=0}^{2\pi} d\theta -$$

$$- \iint \left\{\left[\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 f}{\partial \chi^2} + \lambda\sin\theta f\right]\right\}\delta f\,d\theta d\chi. \tag{4}$$

The integrated parts vanish and the condition that the integral is stationary for general variations δf is

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial\chi^2}+\lambda f=0, \tag{5}$$

which is the differential equation satisfied by S_n provided that $\lambda = n(n+1)$. Then, by the usual argument, our estimate of the irregularity of f is stationary for small variations of f provided f is a surface harmonic of integral degree n, and the estimate of irregularity is then $n(n+1)$.

It is sometimes said that spherical harmonics are an artificial method of analysis, and therefore it seems desirable to point out that they have a direct relation to the property of smoothness when this is defined in what seems to be the most natural way.

It may be added that if K_m, K_n are any orthogonal harmonics,

$$\iint \frac{\partial K_m}{\partial x_i}\frac{\partial K_n}{\partial x_i}\,dS=0. \tag{6}$$

Thus for a general f there are theorems similar to those that hold for expansions in orthogonal functions. A consequence is that if the expansion of f in harmonics contains no harmonics of degrees less than n, the irregularity of f is at least $n(n+1)$.

A problem in geodesy is the interpolation of gravity from irregularly distributed observed values. This result shows that the smoothest interpolate is that given by fitting surface harmonics up to a given degree; this also takes account of the fact that five of the lower harmonics are theoretically absent.

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NOTE ON THE IMPROVEMENT OF APPROXIMATE LATENT ROOTS AND MODAL COLUMNS OF A SYMMETRICAL MATRIX

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SUMMARY

The method given below for the improvement of an approximate set of latent roots and modal columns of a symmetrical matrix is essentially the same as Jahn's *second* method described in (**1**), but the proof is here set out entirely in matrix notation.

The present paper is, in fact, the counterpart of the notes by Collar (**2**) relating to Jahn's *first* method.

1. If M is a modal matrix of a symmetrical matrix A, both $M'AM$ and $M'M$ are diagonals. Thus, if m is an approximation to M, $m'Am$ and $m'm$ will be approximately diagonal, and we can make use of these products to obtain a better approximation to M, as follows,

Let $m'Am = \Lambda_1+a$ and $m'm = \Lambda_2+b$, where Λ_1 and Λ_2 are the diagonals, and a and b the non-diagonal parts, of these matrices. Further, let $M = m(I+\mu)$. By suitable post-multiplication by a diagonal matrix we can always ensure that $I+\mu$ has unit diagonal elements. Since such post-multiplication of M leaves it still a modal matrix of A, we can suppose without loss of generality that μ has no diagonal elements.

Then $(I+\mu')(\Lambda_1+a)(I+\mu)$ and $(I+\mu')(\Lambda_2+b)(I+\mu)$ are diagonals.

Expanding these products, and neglecting second- and third-order terms, we see that $\Lambda_1+(\mu'\Lambda_1+\Lambda_1\mu+a)$ and $\Lambda_2+(\mu'\Lambda_2+\Lambda_2\mu+b)$ must be diagonal matrices. But the items in the brackets have no diagonal terms, and so

$$\mu'\Lambda_1+\Lambda_1\mu+a = \mu'\Lambda_2+\Lambda_2\mu+b = 0,$$

whence
$$\Lambda_1\mu\Lambda_2-\Lambda_2\mu\Lambda_1 = b\Lambda_1-a\Lambda_2. \tag{1}$$

If the term in the rth row and the sth column of μ is μ_{rs}, the corresponding term in $(\Lambda_1\mu\Lambda_2-\Lambda_2\mu\Lambda_1)$ is $(\lambda_{1,r}\lambda_{2,s}-\lambda_{2,r}\lambda_{1,s})\mu_{rs}$, where the λ_1's and λ_2's are respectively the elements of Λ_1 and Λ_2. Hence μ_{rs} is derived from the corresponding term in $(b\Lambda_1-a\Lambda_2)$ by dividing by $(\lambda_{1,r}\lambda_{2,s}-\lambda_{2,r}\lambda_{1,s})$.

Since the latent roots of A are Λ_1/Λ_2 (with errors of order μ^2), the method will not work if two or more roots are equal, since the corresponding divisors then vanish, and it is clearly not very effective if any of the latent

roots are close together. In general, however, the method is a very attractive one, inasmuch as it produces an all-round improvement of the modal matrix in one operation.

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VIBRATION OF CERTAIN SQUARE PLATES HAVING SIMILAR ADJACENT EDGES

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SUMMARY

The fundamental frequencies of flexural vibration are determined for thin uniform elastic square plates that have two adjacent edges either clamped or simply supported and that always have the other two adjacent edges free; a pin-point support exists at the intersection of the free edges. Finite difference methods, which simplify the treatment of the free boundaries for definite values of Poisson's ratio, are used in conjunction with extrapolation procedures to obtain the approximate solutions.

An approximate solution is presented for the fundamental frequency of flexural vibration of a uniform elastic square plate that has two adjacent edges clamped and the other two adjacent edges free; the plate is point supported at the intersection of the free edges so that this point cannot deflect. The fundamental frequency of an otherwise similar square plate which has simply supported adjacent edges is obtained also. (See Fig. 1.)

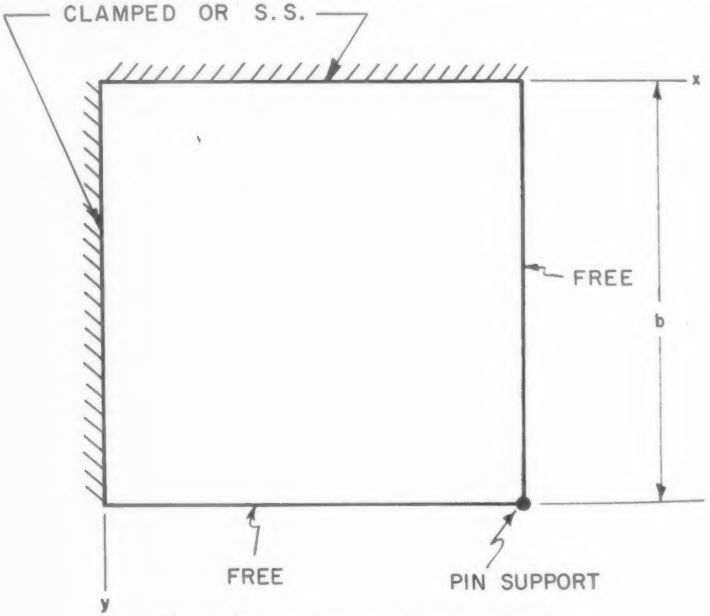

FIG. 1. Types of square plates considered.

The solutions are valid for plates which have a value of Poisson's ratio of 0·3.

The usual assumptions regarding the bending of thin elastic plates are made. If the static lateral loads on the plate are replaced with equivalent inertia forces when the plate is vibrating freely, the governing differential equation for the plate is

$$\nabla^2\nabla^2 w - \frac{q\omega_n^2}{gD} w = 0, \tag{1}$$

where w is the plate deflexion, q is the lateral weight per length squared of the plate, D is the plate stiffness, g is gravitational acceleration, and ω_n is a natural frequency in radians per second. Equation (1) may be written in finite difference form at a selected number of points. First-order operators, which include Poisson's ratio, for points on and adjacent to the free boundaries have been tabulated by Jensen (**1**). If equation (1) is written in difference form, the resulting simultaneous linear algebraic equations may be written in matrix form as

$$[f^4 A - \lambda I]W = 0, \tag{2}$$

where f is the plate width b divided by the grid spacing, A is a matrix of coefficients of the $\nabla^2\nabla^2 w$ terms in (1), I is a unit matrix, W is a column matrix of deflexions, and

$$\lambda = \frac{q\omega_1^2 b^4}{gD}. \tag{3}$$

Equation (2) is in standard eigenvalue form and the lowest root λ may be determined rapidly on an electronic digital computer. Equation (3) may be rewritten as

$$\omega_1 = \frac{k}{b^2}\left(\frac{gD}{q}\right)^{\frac{1}{2}}, \tag{4}$$

where $k = \lambda^{\frac{1}{2}}$. For the case of a square plate having two adjacent edges clamped, values of k have been found to be

$$k = 12{\cdot}89906 \quad \text{for } f = 6,$$

$$k = 12{\cdot}55425 \quad \text{for } f = 5.$$

Values of k for other values of f indicate that k is approaching the true value monotonically; consequently, extrapolation (**2**) yields

$$k = \frac{6^2}{6^2-5^2}12{\cdot}89906 - \frac{5^2}{6^2-5^2}12{\cdot}55425 = 13{\cdot}683. \tag{5}$$

Thus,
$$\omega_1 = \frac{13{\cdot}683}{b^2}\left(\frac{gD}{q}\right)^{\frac{1}{2}}. \tag{6}$$

The above result probably deviates from the true answer by less than one

per cent. Extrapolation gives very accurate results (**2**). The fundamental mode shape is symmetric about a diagonal line passing through the point supported corner. If the pin support at $x = y = b$ is removed, the coefficient k for (4) has been found by Young (**3**) to be 6·958. It is seen that the post support at the corner $x = y = b$ roughly doubles the fundamental frequency of a square plate having two adjacent edges clamped and the other edges free. It is interesting to note that the second mode of the plate in Fig. 1 will have the same frequency as the plate discussed by Young since both plates have a nodal line along the line $x = y$ for the second mode. The value of k for the second mode (**3**) is 24·08.

For the case of a square plate having two adjacent simply supported edges, values of k have been found to be

$$k = 8{\cdot}501848 \quad \text{for } f = 5,$$
$$k = 8{\cdot}222674 \quad \text{for } f = 4.$$

Extrapolation yields

$$k = \frac{5^2}{5^2-4^2}8{\cdot}501848 - \frac{4^2}{5^2-4^2}8{\cdot}222674 = 8{\cdot}998. \tag{7}$$

Thus,
$$\omega_1 = \frac{8{\cdot}998}{b^2}\left(\frac{gD}{q}\right)^{\frac{1}{2}}. \tag{8}$$

The above answer probably deviates from the true answer by less than one per cent.

If the plate in Fig. 1 rests on a uniform elastic foundation (**4**),

$$\omega_1 = \frac{13{\cdot}683}{b^2}\left(\frac{gD}{q}\right)^{\frac{1}{2}} + \left(\frac{gK}{q}\right)^{\frac{1}{2}} \tag{9}$$

for the plate having two adjacent clamped edges and

$$\omega_1 = \frac{8{\cdot}998}{b^2}\left(\frac{gD}{q}\right)^{\frac{1}{2}} + \left(\frac{gK}{q}\right)^{\frac{1}{2}} \tag{10}$$

for the plate having two adjacent simply supported edges. The symbol K represents the elastic foundation modulus per length cubed.

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TABLES OF BENNETT FUNCTIONS FOR THE TWO-FREQUENCY MODULATION PRODUCT PROBLEM FOR THE HALF-WAVE SQUARE-LAW RECTIFIER

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SUMMARY

The first few Bennett functions of the second kind, i.e. coefficients $A^{(2)}_{mn}(k)$ in the double Fourier series expansion of the output from a half-wave square-law rectifier with a two-frequency input, are tabulated as functions of the input amplitude ratio k. Bennett functions $A^{(\nu)}_{mn}(k)$ of the νth kind for the two-frequency modulation product problem for the half-wave νth-law rectifier are defined and recurrence formulae among the $A^{(\nu)}_{mn}(k)$ of given kind and different orders, or among those of different kinds and given orders, are established for use with the tables. Some brief comments are made concerning other forms of the problem and of Bennett functions.

1. Introduction

THE output from a half-wave square-law rectifier with a two-frequency input of amplitudes P and Pk, $0 < k \leqslant 1$, can be expressed as a double Fourier series in the form

$$y(t) = \tfrac{1}{2}P^2 A^{(2)}_{00}(k) + P^2 \sum_{m,n=0}^{\infty}{}^{*} A^{(2)}_{\pm mn}(k)\cos(\omega_{\pm mn}t + \phi_{\pm mn}), \tag{1.1}$$

where the asterisk indicates that the sum is extended over both signs if $mn \neq 0$ and over only the upper sign if $mn = 0$ with the single exception that the term occurring under the summation sign for $m = n = 0$ is not to be included in the sum. In (1.1), the angular frequencies $\omega_{\pm mn}$ and phase angles $\phi_{\pm mn}$ are particular linear combinations of those associated with the input, while the amplitudes are the functions of k defined as

$$A^{(2)}_{\pm mn}(k) = \frac{2}{\pi^2} \iint\limits_R (\cos u + k\cos v)^2 \cos mu \cos nv \, dudv, \tag{1.2}$$

where $(m, n = 0, 1, 2,...)$ and R is the region of the square $0 \leqslant u, v \leqslant \pi$ in which $\cos u + k\cos v \geqslant 0$. For the details we refer to Bennett (**1**). In this paper we give numerical tables of the first sixteen non-zero functions $A^{(2)}_{mn}(k) = A^{(2)}_{\pm mn}(k)$ as defined by (1.2) for $k = 0{\cdot}02(0{\cdot}02)1{\cdot}0$. The tables given are thus a companion set to the tables given by Sternberg, Shipman, and Thurston (**2**) for the corresponding problem for the half-wave linear rectifier. Each entry in the tables is believed to be accurate to somewhat

better than one unit in the last place given and probably most are rounded correctly.

Following the terminology of (**2**), we refer to the functions $A_{mn}^{(2)}(k)$ defined by (1.2) as Bennett functions of the second kind. Similarly, if the same two-frequency input is applied to a half-wave νth-law rectifier, then in place of (1.1) we have for the output the double Fourier series expansion

$$y(t) = \tfrac{1}{2}P^{\nu}A_{00}^{(\nu)}(k)+P^{\nu}\sum_{m,n=0}^{\infty}{}^{*}A^{(\nu)}_{\pm mn}(k)\cos(\omega_{\pm mn}t+\phi_{\pm mn}), \tag{1.3}$$

and in place of (1.2) we have the Bennett functions of the νth kind defined as

$$A^{(\nu)}_{\pm mn}(k) = \frac{2}{\pi^2}\iint_R (\cos u+k\cos v)^{\nu}\cos mu\cos nv\,dudv, \tag{1.4}$$

where ($\nu = 1, 2,...; m, n = 0, 1, 2,...$) and again R is the region of the square $0 \leqslant u,\ v \leqslant \pi$ in which $\cos u+k\cos v \geqslant 0$. Recurrence formulae for the functions $A_{mn}^{(\nu)}(k) = A^{(\nu)}_{\pm mn}(k)$ by means of which the higher-order functions of the νth kind may be expressed in terms of the lower-order functions of the νth kind are derived, together with a set of recurrence identities between the functions $A_{mn}^{(\nu+1)}(k)$ and $A_{m'n'}^{(\nu)}(k)$ of orders $m+n$ and $m'+n' = m+n\pm 1$ which make it possible to express similarly the $(\nu+1)$th-kind functions in terms of those of the νth kind. The tables of the second-kind functions given are thus readily extensible both in order and in kind. In the same manner, the functions $A_{mn}^{(2)}(k)$ tabulated in the present paper may be obtained from the functions $A_{mn}(k) = A_{mn}^{(1)}(k)$ tabulated by Sternberg, Shipman, and Thurston (**2**) by use of these two sets of recurrence formulae. But, owing to round-off errors and loss of order in this process, it is not entirely satisfactory; and taking into consideration the great importance of the special case of the square-law rectifier, it was decided to tabulate the functions $A_{mn}^{(2)}(k)$ separately.

The construction of the tables is described briefly in section 2 and some remarks on the use of the tables, together with the aforementioned recurrence identities, are given in section 3. A proof of the latter relations is presented in section 4, and in section 5 we conclude with some brief remarks concerning various other forms in which the present problem or Bennett functions may appear, in addition to those treated.

2. Construction of the tables

The construction of the tables was carried out on an I.B.M. Card Programmed Calculator by computational methods similar to those described in (**2**) for the functions $A_{mn}(k) = A_{mn}^{(1)}(k)$ and it will not be discussed in detail. We note, however, for reference the series and formulae employed.

Thus, firstly, for all values of k the non-zero even-order functions $A_{mn}^{(2)}(k)$ were computed from the rational formulae

$$\tfrac{1}{2}A_{00}^{(2)}(k) = \tfrac{1}{4}+\tfrac{1}{4}k^2, \qquad A_{20}^{(2)}(k) = \tfrac{1}{4}, \tag{2.1}$$

$$A_{11}^{(2)}(k) = \tfrac{1}{2}k, \qquad A_{02}^{(2)}(k) = \tfrac{1}{4}k^2. \tag{2.2}$$

For $0{\cdot}02 \leqslant k \leqslant 0{\cdot}40$ the odd-order functions $A_{mn}^{(2)}(k)$ were evaluated from the series

$$A_{10}^{(2)}(k) = (1/\pi)[(4/3)+k^2-(1/16)k^4-...], \tag{2.3}$$

$$A_{01}^{(2)}(k) = (k/\pi)[2+(1/4)k^2+(1/96)k^4+...], \tag{2.4}$$

$$A_{30}^{(2)}(k) = (1/\pi)[(4/15)-(1/3)k^2+(3/16)k^4-...], \tag{2.5}$$

$$A_{21}^{(2)}(k) = (k/\pi)[(2/3)-(1/4)k^2+(1/32)k^4+...], \tag{2.6}$$

$$A_{12}^{(2)}(k) = (k^2/\pi)[(1/2)-(1/24)k^2-(1/256)k^4-...], \tag{2.7}$$

$$A_{03}^{(2)}(k) = (k^3/\pi)[(1/12)+(1/192)k^2+(3/2560)k^4+...], \tag{2.8}$$

$$A_{50}^{(2)}(k) = (1/\pi)[-(4/105)+(1/5)k^2-(5/16)k^4+...], \tag{2.9}$$

$$A_{41}^{(2)}(k) = (k/\pi)[-(2/15)+(1/4)k^2-(5/32)k^4+...], \tag{2.10}$$

$$A_{32}^{(2)}(k) = (k^2/\pi)[-(1/6)+(1/8)k^2-(5/256)k^4-...], \tag{2.11}$$

$$A_{23}^{(2)}(k) = (k^3/\pi)[-(1/12)+(1/64)k^2+(1/512)k^4+...], \tag{2.12}$$

$$A_{14}^{(2)}(k) = (k^4/\pi)[-(1/96)-(1/640)k^2-(1/2048)k^4-...], \tag{2.13}$$

$$A_{05}^{(2)}(k) = (k^5/\pi)[(1/960)+(1/2560)k^2+(5/28672)k^4+...], \tag{2.14}$$

while for $0{\cdot}42 \leqslant k \leqslant 0{\cdot}98$ the computation was done from the formulae

$$A_{10}^{(2)}(k) = (8/9\pi^2)[(7+k^2)E(k)-(4-4k^2)K(k)], \tag{2.15}$$

$$A_{01}^{(2)}(k) = (8/9\pi^2k)[(1+7k^2)E(k)-(1+2k^2-3k^4)K(k)], \tag{2.16}$$

$$A_{30}^{(2)}(k) = (8/225\pi^2)[(23-23k^2+8k^4)E(k)+(-8+12k^2-4k^4)K(k)], \tag{2.17}$$

$$A_{21}^{(2)}(k) = (8/45\pi^2k)[(3+7k^2-2k^4)E(k)+(-3+2k^2+k^4)K(k)], \tag{2.18}$$

$$A_{12}^{(2)}(k) = (8/45\pi^2k^2)[(-2+7k^2+3k^4)E(k)+(2-8k^2+6k^4)K(k)], \tag{2.19}$$

$$A_{03}^{(2)}(k) = (8/225\pi^2k^3)[(8-23k^2+23k^4)E(k)+ \\ +(-8+27k^2-34k^4+15k^6)K(k)], \tag{2.20}$$

$$A_{50}^{(2)}(k) = (8/11025\pi^2)[(-281+985k^2-1080k^4+384k^6)E(k)+ \\ +(176-548k^2+564k^4-192k^6)K(k)], \tag{2.21}$$

$$A_{41}^{(2)}(k) = (8/1575\pi^2k)[(15-103k^2+128k^4-48k^6)E(k)+ \\ +(-15+58k^2-67k^4+24k^6)K(k)], \tag{2.22}$$

$$A_{32}^{(2)}(k) = (8/315\pi^2k^2)[(-6+9k^2-19k^4+8k^6)E(k)+ \\ +(6-12k^2+10k^4-4k^6)K(k)], \tag{2.23}$$

$$A_{23}^{(2)}(k) = (8/315\pi^2k^3)[(8-19k^2+9k^4-6k^6)E(k)+ \\ +(-8+23k^2-18k^4+3k^6)K(k)], \tag{2.24}$$

$$A_{14}^{(2)}(k) = (8/1575\pi^2k^4)[(-48+128k^2-103k^4+15k^6)E(k)+ +(48-152k^2+164k^4-60k^6)K(k)], \quad (2.25)$$

$$A_{05}^{(2)}(k) = (8/11025\pi^2k^5)[(384-1080k^2+985k^4-281k^6)E(k)+ +(-384+1272k^2-1501k^4+718k^6-105k^8)K(k)], \quad (2.26)$$

where $K(k)$ and $E(k)$ are the complete elliptic integrals of modulus k as tabulated by Fletcher (**3**). For $k = 1{\cdot}0$ the odd-order functions $A_{mn}^{(2)}(k)$ were computed from the formula

$$A_{mn}^{(2)}(1{\cdot}0) = \frac{64(-1)^{m+1}}{[m^2-n^2][(m+n)^2-4][(m-n)^2-4]\pi^2}, \quad (2.27)$$

where ($m+n = 1, 3, 5,\ldots$). These series and formulae may be derived by methods such as those described in (**1**) and (**2**).

The computation was checked by a number of methods, including differencing, and both the basic calculations and the magnitudes of the differences indicate that each entry in the tables should be accurate to better than 1×10^{-8} units.

3. Applications of the tables

For direct applications of the tables linear interpolation should be generally good to at least four decimal places everywhere. If greater accuracy is desired, a parabolic interpolation formula with second differences may be used to obtain accuracy to about six or more decimal places, or a five-point Lagrange formula is recommended and should give results accurate to about 3×10^{-8} units.

For applications of the tables to problems requiring values of the higher order, or higher-kind functions $A_{mn}^{(\nu)}(k)$, we note the recurrence formulae

$$\begin{aligned}
&(a)\quad (m-n+\nu+2)A_{m+1,n-1}^{(\nu)} \equiv -(m+n-\nu-2)A_{m-1,n-1}^{(\nu)}-2mkA_{mn}^{(\nu)},\\
&(b)\quad (m+n+\nu)A_{mn}^{(\nu)} \equiv -(m-n-\nu-2)A_{m-2,n}^{(\nu)}-2(m-1)kA_{m-1,n-1}^{(\nu)},\\
&(c)\quad (n+m+\nu)A_{mn}^{(\nu)} \equiv -(n-m-\nu-2)A_{m,n-2}^{(\nu)}-2(n-1)(1/k)A_{m-1,n-1}^{(\nu)},\\
&(d)\quad (n-m+\nu+2)A_{m-1,n+1}^{(\nu)} \equiv -(n+m-\nu-2)A_{m-1,n-1}^{(\nu)}-2n(1/k)A_{mn}^{(\nu)},
\end{aligned} \quad (3.1)$$

for functions of the same kind, and the recurrence formulae

$$\begin{aligned}
&(a)\quad A_{00}^{(\nu+1)} \equiv A_{10}^{(\nu)}+kA_{01}^{(\nu)},\\
&(b)\quad 2mA_{mn}^{(\nu+1)} \equiv (\nu+1)A_{m-1,n}^{(\nu)}-(\nu+1)A_{m+1,n}^{(\nu)},\\
&(c)\quad 2nA_{mn}^{(\nu+1)} \equiv k(\nu+1)A_{m,n-1}^{(\nu)}-k(\nu+1)A_{m,n+1}^{(\nu)},
\end{aligned} \quad (3.2)$$

for functions of different kinds, where in (3.1) we have ($\nu, m, n = 1, 2,\ldots;$ $m \neq 1$ in (b); $n \neq 1$ in (c)) and in (3.2) we have ($\nu = 1, 2,\ldots; m, n = 0, 1, 2,\ldots;$

$m \neq 0$ in (b); $n \neq 0$ in (c)). We note in passing that for $\nu = 1$ the relations (3.1) reduce to the corresponding recurrence relations of (**2**).

The application of the relations (3.1) and (3.2) is elementary, and with the aid of the tables one may readily compute values of both the higher-order and higher-kind functions $A_{mn}^{(\nu)}(k)$. It should be noted in this connexion, however, that owing to round-off errors there will generally be a loss of accuracy when k is small in the use of relations (3.1) (c) and (d), while for self-evident reasons there will always be a loss of order in the use of relations (3.2).

4. Proof of the recurrence formulae

To establish the recurrence formulae (3.1) and (3.2) we begin by noting that the integrals (1.4) defining the functions $A_{mn}^{(\nu)}(k)$ may always be written in either of the forms

$$\left.\begin{aligned} A_{mn}^{(\nu)}(k) &= \frac{2}{\pi^2}\int\limits_0^{\pi} R_m^{(\nu)}(v)\cos nv\,dv, \\ A_{mn}^{(\nu)}(k) &= \frac{2}{\pi^2}\int\limits_{\alpha}^{\beta} S_n^{(\nu)}(u)\cos mu\,du+\frac{2}{\pi^2}I_{mn}^{(\nu)}, \end{aligned}\right\} \tag{4.1}$$

where $R_m^{(\nu)}(v)$, $S_n^{(\nu)}(u)$, and $I_{mn}^{(\nu)}$ are the integrals

$$\left.\begin{aligned} R_m^{(\nu)}(v) &= \int\limits_0^{c(v)} (\cos u+k\cos v)^{\nu}\cos mu\,du, \\ S_n^{(\nu)}(u) &= \int\limits_0^{\gamma(u)} (\cos u+k\cos v)^{\nu}\cos nv\,dv, \end{aligned}\right\} \tag{4.2}$$

$$I_{mn}^{(\nu)} = \int\limits_0^{\alpha}\left[\int\limits_0^{\pi} (\cos u+k\cos v)^{\nu}\cos nv\,dv\right]\cos mu\,du, \tag{4.3}$$

and where the limits of integration are $\alpha = \cos^{-1}k$, $\beta = \cos^{-1}(-k)$,

$$c(v) = \cos^{-1}(-k\cos v), \quad \text{and} \quad \gamma(u) = \cos^{-1}(-(1/k)\cos u),$$

while in each of these formulae ($\nu = 1, 2,\ldots;\ m, n = 0, 1, 2,\ldots$). Next we note by direct substitution and simplification using (4.2) the preliminary identities

$$\left.\begin{aligned} (m+\nu+1)R_{m+1}^{(\nu)}(v)+2mkR_m^{(\nu)}(v)\cos v+(m-\nu-1)R_{m-1}^{(\nu)}(v) &\equiv 0, \\ (n+\nu+1)S_{n+1}^{(\nu)}(u)+2n(1/k)S_n^{(\nu)}(u)\cos u+(n-\nu-1)S_{n-1}^{(\nu)}(u) &\equiv 0, \end{aligned}\right\} \tag{4.4}$$

where (ν, m, $n = 1, 2,\ldots$). Multiplying through the first of the relations (4.4) by unity or $\cos nv$ and the second of relations (4.4) by unity or $\cos mu$

and integrating the resulting identities, we obtain, after simplification with the aid of (4.1) and (4.3), the further identities

$$\begin{aligned}
&(a)\quad (m+\nu+1)A^{(\nu)}_{m+1,n}+mkA^{(\nu)}_{m,n-1}+\\
&\qquad\qquad +mkA^{(\nu)}_{m,n+1}+(m-\nu-1)A^{(\nu)}_{m-1,n}\equiv 0,\\
&(d)\quad (n+\nu+1)A^{(\nu)}_{m,n+1}+n(1/k)A^{(\nu)}_{m+1,n}+\\
&\qquad\qquad +n(1/k)A^{(\nu)}_{m-1,n}+(n-\nu-1)A^{(\nu)}_{m,n-1}\equiv 0,
\end{aligned}\tag{4.5}$$

where ($\nu = 1, 2,...$; $m, n = 0, 1, 2,...$; $m \neq 0$ in (a); $n \neq 0$ in (d)). The recurrence formulae (3.1) (a) for $n = 1$ and (3.1) (d) for $m = 1$ now follow at once by taking $n = 0$ or $m = 0$ in the corresponding relation (4.5) while the proof of the remaining recurrence formulae (3.1) (a), (b), (c), and (d) may be completed readily by eliminating in turn $A^{(\nu)}_{m,n-1}(k)$, $A^{(\nu)}_{m,n+1}(k)$, $A^{(\nu)}_{m+1,n}(k)$, and $A^{(\nu)}_{m-1,n}(k)$ from the two relations (4.5) and renumbering the subscripts appropriately.

The first of the recurrence formulae (3.2) is a ready consequence of the definition (1.4) of the functions $A^{(\nu)}_{mn}(k)$. To complete the verification of the remaining two recurrence formulae (3.2) (b) and (c), use (4.2) and (4.4) to derive the preliminary identities

$$\begin{aligned}
2mR_m^{(\nu+1)}(v) &\equiv (\nu+1)R^{(\nu)}_{m-1}(v)-(\nu+1)R^{(\nu)}_{m+1}(v),\\
2nS_n^{(\nu+1)}(u) &\equiv k(\nu+1)S^{(\nu)}_{n-1}(u)-k(\nu+1)S^{(\nu)}_{n+1}(u),
\end{aligned}\tag{4.6}$$

where ($\nu, m, n = 1, 2,...$). Multiplying the first of relations (4.6) by unity or $\cos nv$ and the second of relations (4.6) by unity or $\cos mu$ and integrating the resulting identities, we obtain with the aid of (4.1) and (4.3) the remaining recurrence formulae (3.2) (b) and (c). This completes the proofs.

We note in passing, for reference, the additional identity

$$A^{(\nu+1)}_{mn} \equiv \tfrac{1}{2}A^{(\nu)}_{m+1,n}+\tfrac{1}{2}A^{(\nu)}_{m-1,n}+\tfrac{1}{2}kA^{(\nu)}_{m,n+1}+\tfrac{1}{2}kA^{(\nu)}_{m,n-1}, \tag{4.7}$$

where ($\nu = 1, 2,...$; $m, n = 0, 1, 2,...$). This may be considered as alternative to the identities (3.2). It follows at once from the definition (1.4) of the functions $A^{(\nu)}_{mn}(k)$ by splitting off one factor from the kernel function of the integral $A^{(\nu+1)}_{mn}(k)$ and rearranging the integrand before performing the integration.

5. Other aspects of the problem

In closing this paper it seems worth while to call the reader's attention to several variants of form in which the present problem or Bennett functions may appear. Thus, to cite two examples, we note, on the one hand, that in treating problems of the type here considered several writers have arrived at answers in the form of certain Weber–Schafheitlin integrals which seem in several instances to be completely expressible in terms of

tabulated Bennett functions. See, for instance, formulae (27) and (33) of Bennett (**1**). On the other hand, in treating related but different problems of a statistical nature several authors have derived certain desired correlation functions in forms proportional to what are now tabulated Bennett functions without, of course, anticipating the fact. See, for example, formulae (73) and (74) of Chapter 3 of Lawson and Uhlenbeck (**4**) and formulae (7.14), (7.15), (7.16), and (7.17) of Middleton (**5**). It is an attractive thought that still further applications of these tables and of Bennett functions may yet be found.†

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Tables of the Bennett Functions $A_{mn}^{(2)}(k)$

k	$\frac{1}{2}A_{00}^{(2)}(k)$	$A_{10}^{(2)}(k)$	$A_{01}^{(2)}(k)$	$A_{20}^{(2)}(k)$	$A_{11}^{(2)}(k)$	$A_{02}^{(2)}(k)$
0·02	0·2501	0·4245 4050	0·0127 3303	0·25	0·01	0·0001
·04	·2504	·4249 2243	·0254 6988	·25	·02	·0004
·06	·2509	·4255 5884	·0382 1438	·25	·03	·0009
·08	·2516	·4264 4955	·0509 7034	·25	·04	·0016
·10	·2525	·4275 9429	·0637 4159	·25	·05	·0025
·12	·2536	·4289 9271	·0765 3197	·25	·06	·0036
·14	·2549	·4306 4440	·0893 4531	·25	·07	·0049
·16	·2564	·4325 4885	·1021 8546	·25	·08	·0064
·18	·2581	·4347 0548	·1150 5629	·25	·09	·0081
·20	·2600	·4371 1364	·1279 6164	·25	·10	·0100
·22	·2621	·4397 7259	·1409 0542	·25	·11	·0121
·24	·2644	·4426 8151	·1538 9149	·25	·12	·0144
·26	·2669	·4458 3950	·1669 2379	·25	·13	·0169
·28	·2696	·4492 4558	·1800 0621	·25	·14	·0196
·30	·2725	·4528 9869	·1931 4272	·25	·15	·0225
·32	·2756	·4567 9768	·2063 3727	·25	·16	·0256
·34	·2789	·4609 4132	·2195 9384	·25	·17	·0289
·36	·2824	·4653 2827	·2329 1644	·25	·18	·0324
·38	·2861	·4699 5713	·2463 0911	·25	·19	·0361
·40	·2900	·4748 2641	·2597 7589	·25	·20	·0400
·42	·2941	·4799 3449	·2733 2089	·25	·21	·0441
·44	·2984	·4852 7970	·2869 4822	·25	·22	·0484
·46	·3029	·4908 6024	·3006 6205	·25	·23	·0529
·48	·3076	·4966 7423	·3144 6655	·25	·24	·0576
·50	·3125	·5027 1968	·3283 6597	·25	·25	·0625
·52	·3176	·5089 9450	·3423 6458	·25	·26	·0676
·54	·3229	·5154 9647	·3564 6670	·25	·27	·0729
·56	·3284	·5222 2328	·3706 7671	·25	·28	·0784
·58	·3341	·5291 7250	·3849 9903	·25	·29	·0841
·60	·3400	·5363 4157	·3994 3814	·25	·30	·0900
·62	·3461	·5437 2781	·4139 9860	·25	·31	·0961
·64	·3524	·5513 2840	·4286 8501	·25	·32	·1024
·66	·3589	·5591 4039	·4435 0207	·25	·33	·1089
·68	·3656	·5671 6067	·4584 5455	·25	·34	·1156
·70	·3725	·5753 8599	·4735 4731	·25	·35	·1225
·72	·3796	·5838 1294	·4887 8529	·25	·36	·1296
·74	·3869	·5924 3791	·5041 7356	·25	·37	·1369
·76	·3944	·6012 5714	·5197 1730	·25	·38	·1444
·78	·4021	·6102 6663	·5354 2181	·25	·39	·1521
·80	·4100	·6194 6219	·5512 9255	·25	·40	·1600
·82	·4181	·6288 3937	·5673 3513	·25	·41	·1681
·84	·4264	·6383 9348	·5835 5537	·25	·42	·1764
·86	·4349	·6481 1952	·5999 5928	·25	·43	·1849
·88	·4436	·6580 1215	·6165 5313	·25	·44	·1936
·90	·4525	·6680 6565	·6333 4351	·25	·45	·2025
·92	·4616	·6782 7385	·6503 3733	·25	·46	·2116
·94	·4709	·6886 3001	·6675 4201	·25	·47	·2209
·96	·4804	·6991 2669	·6849 6554	·25	·48	·2304
·98	·4901	·7097 5544	·7026 1681	·25	·49	·2401
1·00	·5000	·7205 0619	·7205 0619	·25	·50	·2500

Tables of the Bennett Functions $A^{(2)}_{mn}(k)$

k	$A^{(2)}_{30}(k)$	$A^{(2)}_{21}(k)$	$A^{(2)}_{12}(k)$	$A^{(2)}_{03}(k)$
0·02	0·0848 4020	0·0042 4350	0·0000 6366	0·0000 0021
·04	·0847 1302	·0084 8317	·0002 5461	·0000 0170
·06	·0845 0144	·0127 1521	·0005 7279	·0000 0573
·08	·0842 0602	·0169 3582	·0010 1805	·0000 1359
·10	·0838 2756	·0211 4118	·0015 9022	·0000 2654
·12	·0833 6710	·0253 2753	·0022 8908	·0000 4588
·14	·0828 2588	·0294 9110	·0031 1433	·0000 7288
·16	·0822 0537	·0336 2815	·0040 6565	·0001 0882
·18	·0815 0726	·0377 3498	·0051 4265	·0001 5501
·20	·0807 3346	·0418 0789	·0063 4490	·0002 1274
·22	·0798 8610	·0458 4326	·0076 7189	·0002 8331
·24	·0789 6751	·0498 3747	·0091 2308	·0003 6803
·26	·0779 8021	·0537 8696	·0106 9788	·0004 6822
·28	·0769 2695	·0576 8822	·0123 9561	·0005 8520
·30	·0758 1067	·0615 3779	·0142 1559	·0007 2031
·32	·0746 3448	·0653 3226	·0161 5702	·0008 7489
·34	·0734 0169	·0690 6828	·0182 1910	·0010 5031
·36	·0721 1579	·0727 4259	·0204 0092	·0012 4792
·38	·0707 8043	·0763 5198	·0227 0154	·0014 6911
·40	·0693 9942	·0798 9331	·0251 1995	·0017 1528
·42	·0679 7675	·0833 6354	·0276 5508	·0019 8783
·44	·0665 1652	·0867 5971	·0303 0580	·0022 8820
·46	·0650 2300	·0900 7894	·0330 7089	·0026 1782
·48	·0635 0055	·0933 1848	·0359 4908	·0029 7817
·50	·0619 5368	·0964 7564	·0389 3904	·0033 7074
·52	·0603 8698	·0995 4789	·0420 3933	·0037 9702
·54	·0588 0513	·1025 3278	·0452 4848	·0042 5856
·56	·0572 1291	·1054 2801	·0485 6489	·0047 5692
·58	·0556 1513	·1082 3139	·0519 8693	·0052 9369
·60	·0540 1668	·1109 4091	·0555 1285	·0058 7049
·62	·0524 2245	·1135 5466	·0591 4082	·0064 8899
·64	·0508 3736	·1160 7094	·0628 6891	·0071 5087
·66	·0492 6632	·1184 8818	·0666 9510	·0078 5787
·68	·0477 1420	·1208 0503	·0706 1725	·0086 1179
·70	·0461 8583	·1230 2031	·0746 3314	·0094 1445
·72	·0446 8595	·1251 3305	·0787 4038	·0102 6774
·74	·0432 1922	·1271 4251	·0829 3650	·0111 7363
·76	·0417 9012	·1290 4821	·0872 1886	·0121 3413
·78	·0404 0299	·1308 4990	·0915 8469	·0131 5134
·80	·0390 6196	·1325 4763	·0960 3104	·0142 2747
·82	·0377 7088	·1341 4176	·1005 5478	·0153 6480
·84	·0365 3331	·1356 3301	·1051 5258	·0165 6576
·86	·0353 5246	·1370 2245	·1098 2088	·0178 3289
·88	·0342 3106	·1383 1160	·1145 5586	·0191 6893
·90	·0331 7135	·1395 0247	·1193 5339	·0205 7680
·92	·0321 7492	·1405 9762	·1242 0895	·0220 5969
·94	·0312 4256	·1416 0032	·1291 1758	·0236 2110
·96	·0303 7407	·1425 1468	·1340 7370	·0252 6502
·98	·0295 6786	·1433 4595	·1390 7089	·0269 9610
1·00	·0288 2025	·1441 0124	·1441 0124	·0288 2025

Tables of the Bennett Functions $A^{(2)}_{mn}(k)$

k	$A^{(2)}_{50}(k)$	$A^{(2)}_{41}(k)$	$A^{(2)}_{32}(k)$
0·02	−0·0121 0064	−0·0008 4819	−0·0000 2121
·04	−·0120 2449	−·0016 9256	−·0000 8478
·06	−·0118 9819	−·0025 2933	−·0001 9047
·08	−·0117 2271	−·0033 5472	−·0003 3790
·10	−·0114 9936	−·0041 6505	−·0005 2654
·12	−·0112 2981	−·0049 5668	−·0007 5571
·14	−·0109 1609	−·0057 2609	−·0010 2457
·16	−·0105 6056	−·0064 6986	−·0013 3215
·18	−·0101 6590	−·0071 8469	−·0016 7732
·20	−·0097 3507	−·0078 6747	−·0020 5880
·22	−·0092 7134	−·0085 1520	−·0024 7520
·24	−·0087 7819	−·0091 2511	−·0029 2496
·26	−·0082 5935	−·0096 9460	−·0034 0640
·28	−·0077 1875	−·0102 2129	−·0039 1771
·30	−·0071 6048	−·0107 0306	−·0044 5694
·32	−·0065 8875	−·0111 3799	−·0050 2203
·34	−·0060 0788	−·0115 2445	−·0056 1080
·36	−·0054 2227	−·0118 6107	−·0062 2094
·38	−·0048 3632	−·0121 4677	−·0068 5006
·40	−·0042 5443	−·0123 8076	−·0074 9563
·42	−·0036 8094	−·0125 6255	−·0081 5507
·44	−·0031 2010	−·0126 9199	−·0088 2567
·46	−·0025 7604	−·0127 6922	−·0095 0466
·48	−·0020 5268	−·0127 9473	−·0101 8918
·50	−·0015 5376	−·0127 6934	−·0108 7631
·52	−·0010 8272	−·0126 9421	−·0115 6309
·54	−·0006 4273	−·0125 7084	−·0122 4648
·56	−·0002 3660	−·0124 0105	−·0129 2344
·58	·0001 3325	−·0121 8704	−·0135 9086
·60	·0004 6480	−·0119 3131	−·0142 4566
·62	·0007 5651	−·0116 3669	−·0148 8473
·64	·0010 0732	−·0113 0636	−·0155 0498
·66	·0012 1669	−·0109 4378	−·0161 0336
·68	·0013 8464	−·0105 5270	−·0166 7685
·70	·0015 1176	−·0101 3717	−·0172 2249
·72	·0015 9921	−·0097 0147	−·0177 3743
·74	·0016 4878	−·0092 5011	−·0182 1890
·76	·0016 6283	−·0087 8782	−·0186 6428
·78	·0016 4433	−·0083 1944	−·0190 7109
·80	·0015 9683	−·0078 4997	−·0194 3704
·82	·0015 2442	−·0073 8442	−·0197 6009
·84	·0014 3168	−·0069 2784	−·0200 3845
·86	·0013 2365	−·0064 8517	−·0202 7065
·88	·0012 0568	−·0060 6120	−·0204 5561
·90	·0010 8337	−·0056 6040	−·0205 9268
·92	·0009 6233	−·0052 8685	−·0206 8177
·94	·0008 4800	−·0049 4396	−·0207 2350
·96	·0007 4531	−·0046 3428	−·0207 1932
·98	·0006 5811	−·0043 5901	−·0206 7192
1·00	·0005 8817	−·0041 1718	−·0205 8589

Tables of the Bennett Functions $A^{(2)}_{mn}(k)$

k	$A^{(2)}_{23}(k)$	$A^{(2)}_{14}(k)$	$A^{(2)}_{05}(k)$
0·02	−0·0000 0021	−0·0000 0000	0·0000 0000
·04	−·0000 0170	−·0000 0001	·0000 0000
·06	−·0000 0573	−·0000 0004	·0000 0000
·08	−·0000 1356	−·0000 0014	·0000 0000
·10	−·0000 2648	−·0000 0033	·0000 0000
·12	−·0000 4571	−·0000 0069	·0000 0001
·14	−·0000 7252	−·0000 0128	·0000 0002
·16	−·0001 0813	−·0000 0218	·0000 0004
·18	−·0001 5375	−·0000 0350	·0000 0006
·20	−·0002 1061	−·0000 0534	·0000 0011
·22	−·0002 7987	−·0000 0782	·0000 0017
·24	−·0003 6270	−·0000 1110	·0000 0027
·26	−·0004 6026	−·0000 1531	·0000 0040
·28	−·0005 7365	−·0000 2063	·0000 0059
·30	−·0007 0397	−·0000 2723	·0000 0083
·32	−·0008 5229	−·0000 3532	·0000 0116
·34	−·0010 1964	−·0000 4511	·0000 0158
·36	−·0012 0701	−·0000 5682	·0000 0211
·38	−·0014 1537	−·0000 7071	·0000 0278
·40	−·0016 4565	−·0000 8703	·0000 0361
·42	−·0018 9873	−·0001 0607	·0000 0464
·44	−·0021 7545	−·0001 2812	·0000 0590
·46	−·0024 7659	−·0001 5351	·0000 0743
·48	−·0028 0289	−·0001 8258	·0000 0926
·50	−·0031 5505	−·0002 1568	·0000 1146
·52	−·0035 3369	−·0002 5321	·0000 1406
·54	−·0039 3936	−·0002 9555	·0000 1714
·56	−·0043 7258	−·0003 4316	·0000 2077
·58	−·0048 3377	−·0003 9648	·0000 2501
·60	−·0053 2328	−·0004 5601	·0000 2995
·62	−·0058 4138	−·0005 2226	·0000 3570
·64	−·0063 8827	−·0005 9578	·0000 4234
·66	−·0069 6402	−·0006 7716	·0000 5002
·68	−·0075 6865	−·0007 6702	·0000 5885
·70	−·0082 0202	−·0008 6602	·0000 6899
·72	−·0088 6392	−·0009 7488	·0000 8061
·74	−·0095 5398	−·0010 9436	·0000 9389
·76	−·0102 7172	−·0012 2527	·0001 0906
·78	−·0110 1649	−·0013 6850	·0001 2636
·80	−·0117 8747	−·0015 2500	·0001 4607
·82	−·0125 8369	−·0016 9580	·0001 6851
·84	−·0134 0393	−·0018 8204	·0001 9406
·86	−·0142 4678	−·0020 8495	·0002 2314
·88	−·0151 1052	−·0023 0591	·0002 5627
·90	−·0159 9316	−·0025 4647	·0002 9407
·92	−·0168 9229	−·0028 0837	·0003 3727
·94	−·0178 0508	−·0030 9363	·0003 8681
·96	−·0187 2806	−·0034 0467	·0004 4389
·98	−·0196 5692	−·0037 4448	·0005 1016
1·00	−·0205 8589	−·0041 1718	·0005 8817

TABLES OF TWO FUNCTIONS REQUIRED IN CERTAIN ATTENUATION PROBLEMS

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SUMMARY

Tables are provided of the functions

(i) $R(X,T) = e^{-T}I_0\{(T^2-X^2)^{\frac{1}{2}}\}+2\int\limits_X^T e^{-w}I_0\{(w^2-X^2)^{\frac{1}{2}}\}\,dw,$

(ii) $2\int\limits_X^T R(X,w)\,dw,$

and an account is given of their derivation. The tables may be useful in the study of surges in flow along pipe-lines and in the transmission of electric current along cables.

1. Introduction

THE problem of pressure surges, due to movements of a valve, in a uniform pipe-line conveying viscous liquid was investigated by Binnie (**1**). Two cases were considered: (i) when a valve at the exit is shut instantaneously; (ii) when the valve is closed in such a way that the exit velocity falls uniformly to zero. Non-dimensional expressions were derived for the pressure changes at a section distant x from the exit in terms of time t and a number k characteristic of the pipe-line and the liquid in it. This paper provides tables from which the pressure variation can be readily computed. The analysis is similar to that of surges in electric transmission lines, and identical when the leakage is zero (Carslaw and Jaeger, **2**); the tables may be of use in this case also. It is because of this application to unrelated fields that publication in the common ground of a mathematical journal has been chosen.

In case (i) the solution involves the function $R(\frac{1}{2}kx, \frac{1}{2}kt)$, where

$$R(X,T) = e^{-T}I_0\{(T^2-X^2)^{\frac{1}{2}}\}+2\int\limits_X^T e^{-w}I_0\{(w^2-X^2)^{\frac{1}{2}}\}\,dw. \tag{1.1}$$

An expansion in terms of a series of Bessel functions is derived in section 2, which may be used to evaluate $R(X,T)$ from existing tables of Bessel functions such as *British Association Mathematical Tables*, Volume X (**3**).

This series reduces to a single term at the most important section which is at $X = 0$, where the greatest pressure in the pipe-line is developed.

In case (ii) the solution involves $Q(\frac{1}{2}kx, \frac{1}{2}kt)$, where

$$Q(X, T) = 2\int_X^T R(X, w)\, dw. \tag{1.2}$$

For this case also, a series of Bessel functions is obtained in section 3, and again it reduces to a single term at $X = 0$.

2. Instantaneous closure of the valve. Case (i)

The equations of the earlier paper, Binnie (**1**), will be referred to, and they will be indicated by bold type. It is shown in (**3.13**) that we have to evaluate

$$R(\tfrac{1}{2}kx, \tfrac{1}{2}kt) = e^{-\frac{1}{2}kt}I_0\{\tfrac{1}{2}k(t^2-x^2)^{\frac{1}{2}}\}+k\int_x^t e^{-\frac{1}{2}k\tau}I_0\{\tfrac{1}{2}k(\tau^2-x^2)^{\frac{1}{2}}\}\, d\tau, \tag{2.1}$$

which reduces to $R(X, T)$ on substituting

$$T = \tfrac{1}{2}kt, \qquad X = \tfrac{1}{2}kx, \qquad w = \tfrac{1}{2}k\tau. \tag{2.2}$$

Thus the number of parameters is diminished by one. With the aid of Lommel's expansion given by Watson (**4**, p. 140), the integral in (2.1) may be put in the form

$$S = 2\int_X^T e^{-w}I_0\{(w^2-X^2)^{\frac{1}{2}}\}\, dw = 2\int_X^T \sum_{n=0}^{\infty} \frac{e^{-w}(-1)^n}{n!\,(2w)^n} X^{2n}I_n(w)\, dw. \tag{2.3}$$

But
$$\int \frac{e^{-w}}{w^n} I_n(w)\, dw = -\frac{1}{2n-1}\,\frac{e^{-w}}{w^{n-1}}\{I_{n-1}(w)+I_n(w)\}, \tag{2.4}$$

therefore
$$S = \left[\sum_{n=0}^{\infty} \frac{(-1)^{n+1}X^{2n}}{n!\,2^{n-1}(2n-1)}\,\frac{e^{-w}}{w^{n-1}}\{I_{n-1}(w)+I_n(w)\}\right]_X^T. \tag{2.5}$$

Now
$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}X^n}{n!\,2^n(2n-1)}\{I_{n-1}(X)+I_n(X)\} = e^X. \tag{2.6}$$

This identity, which seems to be new, is established in the Appendix, section 5. Then we have finally

$$\begin{aligned} S &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}X^{2n}}{n!\,2^{n-1}(2n-1)}\,\frac{e^{-T}}{T^{n-1}}\{I_{n-1}(T)+I_n(T)\}-2X \\ &= 2e^{-T}\Big[T\{I_1(T)+I_0(T)\}+\frac{X^2}{2}\{I_0(T)+I_1(T)\}-\frac{X^4}{24T}\{I_1(T)+I_2(T)\}+ \\ &\qquad +\frac{X^6}{240T^2}\{I_2(T)+I_3(T)\}-...\Big]-2X. \end{aligned} \tag{2.7}$$

This result was checked by expanding (2.1), with (2.7) substituted therein, into series of polynomials and comparing their sum with the like expansion of (**3.19**), which is the corresponding result in the earlier paper. It was during this process that (2.6) was discovered. For $X = 0$, (2.7) reduces to one term only.

3. Exit velocity falling uniformly to zero in time t_0. Case (ii)

It appears from (**4.5**) that we must evaluate

$$Q(X,T) = 2\int\limits_X^T R(X,w)\,dw. \tag{3.1}$$

The integral of the first term of R can be directly found from (2.7). That of the second term can be established from (2.7) by means of two integrals. These are the same as (2.4) with $n-1$ replacing n, and

$$\int \frac{e^{-w}}{w^{n-1}} I_n(w)\,dw = \frac{e^{-w}}{w^{n-1}} I_{n-1}(w) - \frac{1}{2n-3}\frac{e^{-w}}{w^{n-2}}\{I_{n-2}(w)+I_{n-1}(w)\}, \tag{3.2}$$

which is obtained by integration by parts. Hence

$$\int \frac{e^{-w}}{w^{n-1}}\{I_{n-1}(w)+I_n(w)\}\,dw = \frac{e^{-w}}{w^{n-2}}\left[\frac{I_{n-1}(w)}{w} - \frac{2}{2n-3}\{I_{n-2}(w)+I_{n-1}(w)\}\right]$$

$$= -\frac{e^{-w}}{w^{n-2}(2n-3)}\left[I_{n-2}(w)+\left(2+\frac{1}{w}\right)I_{n-1}(w)+I_n(w)\right]. \tag{3.3}$$

There is little to choose between these two forms. We also have the identity

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}X^n}{n!\,2^n(2n-1)}\left[\frac{I_{n-1}(X)}{X} - \frac{2}{2n-3}\{I_{n-2}(X)+I_{n-1}(X)\}\right] = \left(1+\frac{X}{3}\right)e^X, \tag{3.4}$$

which is analogous to (2.6) and is proved in the Appendix, section 6. Hence

$$\int\limits_X^T S\,dT = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}X^{2n}}{n!\,2^{n-1}(2n-1)}\left[\frac{e^{-T}}{T^{n-2}}\left(\frac{I_{n-1}(T)}{T} - \frac{2}{2n-3}\{I_{n-2}(T)+I_{n-1}(T)\}\right) - \right.$$

$$\left. - \frac{e^{-X}}{X^{n-2}}\left(\frac{I_{n-1}(X)}{X} - \frac{2}{2n-3}\{I_{n-2}(X)+I_{n-1}(X)\}\right)\right] - 2X(T-X)$$

$$= e^{-T}\left[2T^2\left(\frac{I_1(T)}{T} + \tfrac{2}{3}\{I_2(T)+I_1(T)\}\right) + \right.$$

$$+ X^2T\left(\frac{I_0(T)}{T} + 2\{I_1(T)+I_0(T)\}\right) -$$

$$- \frac{X^4}{12}\left(\frac{I_1(T)}{T} - 2\{I_0(T)+I_1(T)\}\right) +$$

$$\left. + \frac{X^6}{120T}\left(\frac{I_2(T)}{T} - \tfrac{2}{3}\{I_1(T)+I_2(T)\}\right) - \ldots\right] - 2XT - \tfrac{2}{3}X^3. \tag{3.5}$$

Again, at $X = 0$ only one term remains.

4. Description of the tables

The tables were computed at the Mathematics Division of the National Physical Laboratory, mainly under the direction of Mr. G. F. Miller. The table of $R(X,T)$ was compiled by means of the series (2.7) and that of $Q(X,T)$ by quadrature with (3.5) supplying check values and also initial values for $T \simeq X$.

The tables give values of $R(X,T)$ and $Q(X,T)$, each for $X = 0(0{\cdot}2)5$ and $T = 0(0{\cdot}2)5(1)20$. When $T \leqslant 5$, and when $X < 1$, $5 \leqslant T \leqslant 20$, second differences are provided both X-wise and T-wise for interpolation by means of Everett's formula

$$f(a+\theta h) = (1-\theta)f(a)+\theta f(a+h)+E_0''(\theta)\delta^2 f(a)+E_1''(\theta)\delta^2 f(a+h). \quad (4.1)$$

By replacing δ^2 by $(\delta^2-0{\cdot}184\delta^4)$ differences were modified where necessary to allow for the effect of fourth differences. The formula (4.1) thus always suffices to give full accuracy. Differences were rounded off *after* formation and modification so they cannot be exactly reproduced from the values of R and Q given in the tables.

In those parts of the table where differences are not provided, linear interpolation will give an error not exceeding $0{\cdot}003$ in R; the error from linear interpolation in Q may be larger, up to $0{\cdot}15$ near $X = 1$, $T = 20$, with a tendency to be smaller for the smaller values of Q. More accurate interpolation is best done in this part of the table by first forming the missing differences.

As an example of the use of the tables, we will evaluate $R(0{\cdot}45, 3{\cdot}17)$. (With the aid of (**2.6**), (**2.8**), and (2.2) this example can be shown to correspond to time 88·1 sec. at a point halfway along a pipe-line of diameter 6 in. and length 100,000 ft., which is conveying a liquid of kinematic viscosity $5{\cdot}63\times10^{-4}$ ft.²/sec. and in which the wave velocity is 4,000 ft./sec.) First we determine $R(0{\cdot}4, 3{\cdot}17)$ and $R(0{\cdot}6, 3{\cdot}17)$ with

$$\theta = \frac{3{\cdot}17-3{\cdot}0}{0{\cdot}2} = 0{\cdot}85, \qquad 1-\theta = 0{\cdot}15,$$

$$E_0''(0{\cdot}85) = -0{\cdot}0244, \qquad E_1''(0{\cdot}85) = -0{\cdot}0393.$$

Then

$$R(0{\cdot}4, 3{\cdot}17) = 2{\cdot}21939, \qquad \delta^2 = +3132,$$

$$R(0{\cdot}6, 3{\cdot}17) = 1{\cdot}89846, \qquad \delta^2 = +3070,$$

the values of δ^2 being obtained by linear interpolation. Throughout the working a fifth decimal is retained as a guard figure. For the X-wise interpolation

$$\theta = \frac{0{\cdot}45-0{\cdot}4}{0{\cdot}2} = 0{\cdot}25, \qquad 1-\theta = 0{\cdot}75,$$

$$E_0''(0{\cdot}25) = -0{\cdot}0547, \qquad E_1''(0{\cdot}25) = -0{\cdot}0391,$$

whence

$$R(0{\cdot}45, 3{\cdot}17) = 2{\cdot}1362,$$

the guard figure now being rejected. The coefficients $E_0''(\theta)$ and $E_1''(\theta)$ were taken from *Interpolation and Allied Tables* (**5**), pp. 800–801.

APPENDIX

5. Derivation of (2.6)

A proof of the formula (2.6) seems desirable. Consider

$$V = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}X^n}{n!\,2^n(2n-1)}\{I_{n-1}(X)+I_n(X)\}. \tag{5.1}$$

We wish to show that $V = e^X$, and we shall establish this identity by proving that $dV/dX = V$.

Differentiating (5.1) and using the relations

$$\frac{dI_n(X)}{dX} = I_{n-1}(X) - \frac{n}{X}I_n(X), \qquad \frac{dI_{n-1}(X)}{dX} = I_n(X) + \frac{n-1}{X}I_{n-1}(X), \tag{5.2}$$

we obtain

$$\frac{dV}{dX} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}X^n}{n!\,2^n(2n-1)}\left\{\left(1+\frac{2n-1}{X}\right)I_{n-1}(X)+I_n(X)\right\}$$

$$= V + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}X^{n-1}}{n!\,2^n} I_{n-1}(X). \tag{5.3}$$

Now

$$\sum_{n=0}^{\infty} \frac{(-1)^n X^n}{n!\,2^n} I_{n+s}(X) = \frac{X^s}{s!\,2^s}, \quad s \geq 0; \tag{5.4}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n X^n}{n!\,2^n} I_{n-s}(X) = 0, \quad s \geq 1. \tag{5.5}$$

(See Watson (**4**), p. 141, formulae (7) and (9) with $z = iX$; the latter formula is true only when $\nu > 0$, integer or not.) From (5.5) with $s = 1$ we see that the series in (5.3) is zero and, since $V = 1$ when $X = 0$, it follows that

$$V = e^X \tag{5.6}$$

as required.

In view of the error in Watson's formula (9) quoted above, it seems worth while to give a short proof of (5.4) and (5.5) for integral s. The generating series for the Bessel functions $I_s(X)$ is

$$\exp\left\{\tfrac{1}{2}X\left(t+\frac{1}{t}\right)\right\} = \sum_{s=-\infty}^{\infty} I_s(X)t^s,$$

whilst

$$\exp\left\{-\frac{1}{2}\frac{X}{t}\right\} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,2^n}\frac{X^n}{t^n}.$$

Thus

$$\exp\{\tfrac{1}{2}Xt\} = \sum_{s=-\infty}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^n X^n}{n!\,2^n} I_s(X)t^{s-n}, \tag{5.7}$$

and we may equate coefficients of positive or zero powers of t on both sides to obtain (5.4); we may also equate coefficients of negative powers of t to obtain (5.5). All the series converge absolutely for all values of X and t since $I_s(X) \simeq (\tfrac{1}{2}X)^s/s!$ for large s; whilst $I_{-s}(X) = I_s(X)$ when s is an integer.

6. Derivation of (3.4)

This identity may be proved by the method of section 5 applied twice. We write W for the left side of (3.4) and differentiate, obtaining

$$\frac{dW}{dX} = -\sum_{n=0}^{\infty} \frac{(-1)^{n+1}X^n}{n!\,2^n(2n-1)}\left[\frac{2}{2n-3}\{I_{n-2}(X)+I_{n-1}(X)\}+\right.$$

$$\left.+\frac{2n-1}{2n-3}\frac{I_{n-2}(X)}{X}+\frac{2}{2n-3}\frac{I_{n-1}(X)}{X}\right] = W-Z, \tag{6.1}$$

where

$$Z = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}X^{n-1}}{n!\,2^n(2n-3)}\{I_{n-2}(X)+I_{n-1}(X)\}. \tag{6.2}$$

Again

$$\frac{dZ}{dX} = Z+\sum_{n=0}^{\infty} \frac{(-1)^{n+1}X^{n-2}}{n!\,2^n} I_{n-2}(X) = Z, \tag{6.3}$$

the series being zero by (5.5) with $s = 2$; therefore

$$Z = \lambda e^X,$$

and

$$W = (\mu-\lambda X)e^X, \tag{6.4}$$

in which λ and μ are constants. But at $X = 0$, $W = 1$ and $dW/dX = 4/3$, whence (3.4) follows.

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Table of R(X, T)

T	R	δ_T^2	δ_X^2	R	δ_T^2	δ_X^2	R	δ_T^2	δ_X^2	R	δ_T^2	δ_X^2
X	0·0			0·2			0·4			0·6		
0·0	1·0000	−201	+599									
0·2	1·1906	165	562	0·8187	−163	+558						
0·4	1·3647	137	530	0·9912	135	526	0·6703	−129	+514			
0·6	1·5251	115	502	1·1502	113	498	0·8251	108	488	0·5488	−100	+470
0·8	1·6740	98	477	1·2978	96	474	0·9691	92	464	0·6868	85	449
1·0	1·8131	−83	+455	1·4359	−82	+452	1·1039	−79	+444	0·8162	−73	+429
1·2	1·9439	72	436	1·5657	71	433	1·2308	68	425	0·9384	63	412
1·4	2·0675	63	418	1·6884	62	416	1·3509	59	409	1·0543	55	397
1·6	2·1848	55	403	1·8049	54	401	1·4651	52	394	1·1647	48	383
1·8	2·2966	48	389	1·9161	48	387	1·5741	46	380	1·2702	43	370
2·0	2·4036	−43	+376	2·0224	−43	+374	1·6786	−41	+368	1·3715	−38	+359
2·2	2·5063	39	364	2·1245	38	362	1·7789	37	357	1·4690	34	348
2·4	2·6051	35	353	2·2228	34	351	1·8756	33	347	1·5630	31	338
2·6	2·7005	32	343	2·3176	31	342	1·9689	30	337	1·6539	28	330
2·8	2·7926	29	334	2·4093	28	332	2·0593	27	328	1·7421	26	321
3·0	2·8820	−26	+325	2·4982	−26	+324	2·1469	−25	+320	1·8276	−24	+313
3·2	2·9687	24	317	2·5845	24	316	2·2320	23	312	1·9108	22	306
3·4	3·0529	22	310	2·6684	22	309	2·3148	21	305	1·9917	20	299
3·6	3·1350	21	303	2·7501	20	302	2·3955	20	299	2·0707	19	293
3·8	3·2150	19	297	2·8298	19	296	2·4742	18	292	2·1478	17	287
4·0	3·2930	−18	+291	2·9076	−18	+290	2·5510	−17	+287	2·2232	−16	+282
4·2	3·3693	17	285	2·9835	17	284	2·6262	16	281	2·2969	15	276
4·4	3·4439	16	279	3·0579	16	278	2·6997	15	276	2·3691	14	271
4·6	3·5169	15	274	3·1307	15	273	2·7717	14	271	2·4398	14	267
4·8	3·5885	14	269	3·2020	14	269	2·8423	13	266	2·5092	13	262
5·0	3·6587	−13	+265	3·2719	−13	+264	2·9115	−13	+262	2·5773	−12	+258
X	0·8			1·0			1·2			1·4		
0·8	0·4493	−76	+427									
1·0	0·5716	−65	+410	0·3679	−55	+386						
1·2	·6873	56	395	·4756	48	373	0·3012	−39	+347			
1·4	·7974	49	381	·5785	42	360	·3957	34	337	0·2466	−25	+310
1·6	0·9026	43	368	·6772	37	349	·4868	30	328	·3292	23	303
1·8	1·0034	39	357	·7722	33	339	·5749	27	319	·4096	21	296
2·0	1·1004	−35	+346	0·8638	−30	+330	0·6603	−25	+311	0·4879	−19	+290
2·2	1·1939	31	336	0·9524	27	322	·7431	23	304	·5643	17	284
2·4	1·2843	28	327	1·0384	25	314	·8238	21	297	·6389	16	279
2·6	1·3719	26	319	1·1218	23	306	·9023	19	291	·7119	15	274
2·8	1·4569	24	312	1·2030	21	299	0·9790	18	285	·7835	14	269
3·0	1·5396	−22	+304	1·2821	−19	+293	1·0538	−16	+280	0·8536	−13	+264
3·2	1·6201	20	298	1·3593	18	287	1·1271	15	274	0·9224	13	260
3·4	1·6986	19	291	1·4346	17	281	1·1988	14	269	0·9899	12	256
3·6	1·7753	17	286	1·5084	16	276	1·2691	14	265	1·0562	11	252
3·8	1·8502	16	280	1·5805	15	271	1·3380	13	260	1·1215	11	248
4·0	1·9235	−15	+275	1·6513	−14	+266	1·4057	−12	+256	1·1857	−10	+244
4·2	1·9953	14	270	1·7206	13	262	1·4721	11	252	1·2488	10	241
4·4	2·0656	13	265	1·7886	12	257	1·5374	11	248	1·3110	9	238
4·6	2·1346	13	261	1·8555	12	253	1·6017	10	245	1·3723	9	234
4·8	2·2024	12	256	1·9212	11	249	1·6649	10	241	1·4327	8	231
5·0	2·2689	−11	+252	1·9857	−10	+246	1·7271	−9	+238	1·4923	−8	+228

Table of $Q(X, T)$

T	Q	δ_T^2	δ_X^2	Q	δ_T^2	δ_X^2	Q	δ_T^2	δ_X^2	Q	δ_T^2	δ_X^2
	X	0·0		0·2			0·4			0·6		
0·0	0·000	+79	+241									
0·2	0·439	73	264	0·000	+71	+204						
0·4	0·950	67	286	0·362	66	225	0·000	+63	+172			
0·6	1·529	62	306	0·791	61	246	0·299	60	192	0·000	+57	+145
0·8	2·169	58	326	1·281	57	265	0·659	56	211	0·247	53	164
1·0	2·867	+54	+345	1·828	+54	+284	1·074	+52	+229	0·548	+50	+181
1·2	3·618	51	362	2·429	50	301	1·541	49	247	0·899	48	198
1·4	4·421	48	379	3·080	48	318	2·057	47	263	1·298	45	214
1·6	5·271	46	396	3·779	46	335	2·621	45	279	1·742	43	230
1·8	6·168	44	412	4·523	43	350	3·229	43	295	2·229	41	245
2·0	7·108	+42	+427	5·311	+42	+366	3·879	+41	+310	2·758	+40	+260
2·2	8·090	40	442	6·140	40	380	4·571	39	324	3·326	38	274
2·4	9·113	39	456	7·010	39	395	5·302	38	339	3·933	37	287
2·6	10·174	37	470	7·918	37	409	6·071	37	352	4·576	36	301
2·8	11·272	36	484	8·864	36	422	6·877	36	365	5·255	35	314
3·0	12·408	+35	+497	9·845	+35	+435	7·718	+35	+378	5·969	+34	+327
3·2	13·578	34	510	10·862	34	448	8·594	34	391	6·717	33	339
3·4	14·782	33	522	11·912	33	460	9·503	33	403	7·498	32	351
3·6	16·020	32	534	12·996	32	473	10·445	32	416	8·310	31	363
3·8	17·290	32	546	14·112	31	485	11·419	31	427	9·154	30	375
4·0	18·591	+31	+558	15·260	+31	+496	12·425	+30	+439	10·028	+30	+386
4·2	19·924	30	570	16·438	30	508	13·460	30	450	10·932	29	397
4·4	21·287	30	581	17·646	29	519	14·525	29	461	11·865	29	408
4·6	22·679	29	592	18·884	29	530	15·620	29	472	12·827	28	419
4·8	24·100	28	603	20·151	28	541	16·742	28	483	13·817	27	429
5·0	25·550	+28	+614	21·446	+28	+552	17·893	+27	+494	14·835	+27	+440
	X	0·8		1·0			1·2			1·4		
0·8	0·000	+50	+123									
1·0	0·204	+48	+139	0·000	+44	+104						
1·2	0·456	45	156	·169	42	119	0·000	+39	+87			
1·4	0·753	43	171	·380	40	133	·140	37	101	0·000	+34	+73
1·6	1·094	41	186	·631	39	147	·316	36	114	·115	33	86
1·8	1·475	40	201	0·921	37	161	·529	35	127	·263	32	98
2·0	1·896	+38	+215	1·248	+36	+175	0·776	+34	+140	0·443	+31	+109
2·2	2·355	37	228	1·612	35	188	1·056	33	152	0·653	30	121
2·4	2·851	36	241	2·010	34	200	1·370	32	164	0·894	30	132
2·6	3·382	35	254	2·442	33	213	1·715	31	176	1·164	29	143
2·8	3·948	34	267	2·907	32	225	2·092	30	187	1·463	28	154
3·0	4·547	+33	+279	3·404	+31	+237	2·498	+30	+199	1·791	+28	+165
3·2	5·179	32	291	3·933	31	248	2·934	29	210	2·146	27	175
3·4	5·843	31	303	4·491	30	260	3·400	28	221	2·528	27	185
3·6	6·538	30	315	5·080	29	271	3·893	28	231	2·938	26	196
3·8	7·263	30	326	5·698	29	282	4·415	27	242	3·373	26	206
4·0	8·018	+29	+337	6·344	+28	+293	4·963	+27	+252	3·835	+25	+215
4·2	8·801	28	348	7·019	27	303	5·539	26	262	4·322	25	225
4·4	9·614	28	359	7·721	27	313	6·141	26	272	4·834	25	235
4·6	10·454	27	369	8·449	26	324	6·769	25	282	5·370	24	244
4·8	11·321	27	380	9·205	26	334	7·422	25	292	5·931	24	254
5·0	12·216	+26	+390	9·986	+26	+344	8·101	+25	+301	6·516	+24	+263

Table of $R(X, T)$

T	R	δ_T^2	δ_X^2	R	δ_T^2	δ_X^2	R	δ_T^2	δ_X^2	R	δ_T^2	δ_X^2
X	**1·6**			**1·8**			**2·0**			**2·2**		
1·6	0·2019	−15	+276									
1·8	·2739	14	271	0·1653	−6	+245						
2·0	0·3445	−13	+267	0·2278	−6	+242	0·1353	−0	+217			
2·2	·4138	12	263	·2897	6	240	·1895	1	216	0·1108	+5	+191
2·4	·4820	11	259	·3509	6	237	·2435	1	214	·1576	3	191
2·6	·5490	11	255	·4114	6	234	·2974	2	213	·2046	2	191
2·8	·6149	10	251	·4714	6	232	·3511	2	212	·2520	1	191
3·0	0·6797	−10	+248	0·5307	−6	+229	0·4045	−3	+210	0·2994	+1	+191
3·2	·7436	9	244	·5893	6	227	·4577	3	209	·3470	0	190
3·4	·8066	9	241	·6473	6	225	·5105	3	207	·3945	0	190
3·6	·8686	9	238	·7047	6	222	·5631	3	206	·4420	−1	189
3·8	·9298	8	235	·7615	6	220	·6153	4	204	·4894	1	188
4·0	0·9901	−8	+232	0·8177	−6	+218	0·6671	−4	+203	0·5367	−1	+187
4·2	1·0496	8	229	·8733	6	215	·7185	4	201	·5839	2	186
4·4	1·1084	7	226	·9283	6	213	·7696	4	200	·6308	2	186
4·6	1·1664	7	223	0·9828	6	211	·8203	4	198	·6776	2	185
4·8	1·2237	7	221	1·0367	5	209	·8706	4	197	·7242	2	184
5·0	1·2802	−7	+218	1·0900	−5	+207	0·9205	−4	+195	0·7705	−2	+183

T	R	δ_T^2	δ_X^2	R	δ_T^2	δ_X^2	R	δ_T^2	δ_X^2	R	δ_T^2	δ_X	R	δ_T^2	δ_X^2
X	**2·4**			**2·6**			**2·8**			**3·0**			**3·2**		
2·4	*907	+8	+168												
2·6	1310	6	169	*743	+10	+147									
2·8	1719	5	170	1089	9	149	*608	+12	+128						
3·0	2134	+4	+171	1444	+7	+151	906	+10	+131	*498	+12	+112			
3·2	2552	3	171	1807	6	152	1213	9	133	753	11	115	*408	+13	+98
3·4	2974	2	172	2175	5	153	1529	7	135	1018	9	118	626	11	101
3·6	3398	2	172	2548	4	154	1852	6	137	1294	8	120	855	10	104
3·8	3824	1	172	2926	3	155	2182	5	138	1577	7	122	1094	9	106
4·0	4251	+1	+172	3306	+3	+156	2517	+5	+140	1868	+6	+124	1342	+8	+109
4·2	4678	0	171	3690	2	156	2857	4	141	2165	5	125	1598	7	111
4·4	5106	0	171	4075	2	156	3200	3	141	2467	5	127	1860	6	112
4·6	5534	0	171	4462	1	156	3547	3	142	2774	4	128	2129	5	114
4·8	5961	−1	170	4851	1	156	3897	2	143	3086	4	129	2403	5	116
5·0	6388	−1	+170	5240	+1	+156	4249	+2	+143	3401	+3	+130	2682	+4	+117

T	R	δ_T^2	δ_X^2	R	δ_T^2	δ_X^2	R	δ_T^2	δ_X^2	R	δ_T^2	δ_X^2	R	δ_T^2	δ_X^2
X	**3·4**			**3·6**			**3·8**			**4·0**			**4·2**		
3·4	*334	+13	+85												
3·6	520	11	88	*273	+12	+74									
3·8	718	10	91	432	11	77	*224	+12	+64						
4·0	925	+9	+94	602	+10	+80	359	+11	+67	*183	+11	+55			
4·2	1142	8	96	782	9	83	505	10	70	298	10	58	*150	+10	+48
4·4	1366	7	99	971	8	85	661	9	73	424	9	61	248	9	51
4·6	1598	7	101	1168	7	88	825	8	76	558	8	64	355	9	54
4·8	1837	6	102	1372	7	90	998	7	78	701	8	67	471	8	56
5·0	2081	+5	+104	1583	+6	+92	1178	+7	+80	852	+7	+69	596	+7	+59

T	R	δ_T^2	δ_X^2	R	δ_T^2	δ_X^2	R	δ_T^2	δ_X^2	R	δ_T^2	δ_X^2
X	**4·4**			**4·6**			**4·8**			**5·0**		
4·4	*123	+9	+41									
4·6	206	9	44	*101	+9	+35						
4·8	298	8	47	171	8	38	*82	+8	+30			
5·0	398	+8	+49	250	+8	+41	142	+7	+33	*67	+7	+26

* In units of the 4th decimal.

Table of Q(X, T)

T	Q (X = 1·6)	δ^2_T	δ^2_X	Q (X = 1·8)	δ^2_T	δ^2_X	Q (X = 2·0)	δ^2_T	δ^2_X	Q (X = 2·2)	δ^2_T	δ^2_X
1·6	0·000	+29	+62									
1·8	0·095	29	73	0·000	+25	+52						
2·0	0·219	+28	+83	0·079	+25	+62	0·0000	+217	+436			
2·2	·371	27	94	·182	25	71	·0650	216	522	0·0000	+187	+366
2·4	·550	27	104	·310	24	81	·1516	216	608	·0537	188	442
2·6	·756	27	115	·463	24	90	·2597	215	694	·1261	189	519
2·8	0·989	26	125	·639	24	100	·3894	214	779	·2174	190	595
3·0	1·248	+26	+135	0·840	+24	+109	0·5406	+213	+863	0·3277	+190	+672
3·2	1·532	25	145	1·064	23	118	0·7130	212	947	·4569	190	748
3·4	1·843	25	154	1·311	23	127	0·9067	211	1030	·6052	190	824
3·6	2·178	25	164	1·582	23	136	1·1214	209	1113	·7725	190	900
3·8	2·537	24	173	1·875	23	145	1·3571	208	1195	0·9588	189	975
4·0	2·921	+24	+183	2·191	+22	+153	1·6136	+207	+1277	1·1641	+189	+1050
4·2	3·329	24	192	2·529	22	162	1·8907	205	1357	1·3882	188	1125
4·4	3·761	23	201	2·889	22	171	2·1883	204	1438	1·6311	188	1199
4·6	4·216	23	210	3·272	22	179	2·5063	202	1517	1·8928	187	1273
4·8	4·694	23	219	3·675	21	188	2·8445	200	1596	2·1732	186	1347
5·0	5·195	+22	+228	4·101	+21	+196	3·2027	+199	+1674	2·4721	+185	+1420

T	Q (X = 2·4)	δ^2_T	δ^2_X	Q (X = 2·6)	δ^2_T	δ^2_X	Q (X = 2·8)	δ^2_T	δ^2_X	Q (X = 3·0)	δ^2_T	δ^2_X	Q (X = 3·2)	δ^2_T	δ^2_X
2·4	0*	159	308												
2·6	443	162	374	0*	137	257									
2·8	1049	165	442	366	140	317	0*	116	216						
3·0	1819	167	510	873	144	377	302	121	268	0*	100	180			
3·2	2757	168	579	1523	146	437	726	125	321	250	104	226	0*	84	152
3·4	3862	169	647	2319	148	498	1274	128	374	604	108	273	206	90	191
3·6	5136	170	716	3263	150	560	1950	131	429	1066	112	320	502	94	232
3·8	6581	171	785	4358	152	622	2757	133	484	1640	115	369	892	97	274
4·0	8196	171	853	5604	153	684	3696	135	540	2328	118	418	1379	101	317
4·2	9981	171	922	7003	154	746	4771	137	596	3135	120	468	1966	104	361
4·4	11938	171	990	8556	155	809	5982	138	652	4061	122	518	2658	106	406
4·6	14066	171	1059	10264	155	871	7332	139	709	5109	124	569	3456	109	451
4·8	16366	171	1127	12126	156	934	8821	140	766	6281	125	621	4362	111	497
5·0	18835	171	1195	14144	156	996	10450	141	823	7578	127	673	5379	112	543

T	Q (X = 3·4)	δ^2_T	δ^2_X	Q (X = 3·6)	δ^2_T	δ^2_X	Q (X = 3·8)	δ^2_T	δ^2_X	Q (X = 4·0)	δ^2_T	δ^2_X	Q (X = 4·2)	δ^2_T	δ^2_X
3·4	0*	72	127												
3·6	170	77	161	0*	61	106									
3·8	418	81	197	141	66	136	0*	52	89						
4·0	746	85	234	347	70	168	116	56	115	0*	44	74			
4·2	1159	88	272	624	74	200	289	60	142	96	48	97	0*	37	62
4·4	1660	91	311	974	77	234	522	64	171	240	52	121	79	41	82
4·6	2253	94	351	1401	80	268	819	67	201	436	56	146	200	45	102
4·8	2940	97	392	1909	83	304	1183	70	231	688	59	172	365	48	124
5·0	3723	99	433	2500	86	340	1618	73	263	998	62	199	578	51	147

T	Q (X = 4·4)	δ^2_T	δ^2_X	Q (X = 4·6)	δ^2_T	δ^2_X	Q (X = 4·8)	δ^2_T	δ^2_X	Q (X = 5·0)	δ^2_T	δ^2_X
4·4	0*	31	52									
4·6	65	35	69	0*	27	43						
4·8	166	38	87	54	30	58	0*	22	36			
5·0	305	42	106	138	33	74	45	25	49	0*	19	30

* In units of the 4th decimal. All second differences are positive.

Table of $R(X, T)$

T	R	δ^2_T	δ^2_X	R	δ^2_T	δ^2_X	R	δ^2_T	δ^2_X	R	δ^2_T	δ^2_X	R	δ^2_T	δ^2_X
	X 0·0			0·2			0·4			0·6			0·8		
5	3·659	−33	+26	3·272	−33	+26	2·912	−32	+26	2·577	−31	+26	2·269	−29	+25
6	3·991	26	24	3·603	25	24	3·240	25	24	2·901	24	24	2·586	22	23
7	4·298	20	23	3·910	20	23	3·544	20	23	3·201	19	22	2·880	18	22
8	4·585	17	22	4·195	17	21	3·828	16	21	3·481	16	21	3·156	15	21
9	4·854	14	20	4·464	14	20	4·095	14	20	3·746	13	20	3·417	13	20
10	5·110	−12	+19	4·719	−12	+19	4·349	−12	+19	3·997	−12	+19	3·665	−11	+19
11	5·353	11	19	4·962	11	19	4·590	10	18	4·237	10	18	3·901	10	18
12	5·586	9	18	5·195	9	18	4·821	9	18	4·466	9	18	4·128	9	17
13	5·809	8	17	5·418	8	17	5·044	8	17	4·686	8	17	4·346	8	17
14	6·024	7	17	5·633	7	17	5·258	7	17	4·899	7	16	4·557	7	16
15	6·232	−7	+16	5·840	−7	+16	5·464	−7	+16	5·104	−6	+16	4·760	−6	+16
16	6·433	6	16	6·041	6	16	5·664	6	16	5·303	6	15	4·957	6	15
17	6·628	6	15	6·236	6	15	5·858	5	15	5·496	5	15	5·149	5	15
18	6·817	5	15	6·425	5	15	6·047	5	15	5·684	5	15	5·335	5	14
19	7·002	5	14	6·609	5	14	6·230	5	14	5·866	5	14	5·516	5	14
20	7·181	−4	+14	6·788	−4	+14	6·409	−4	+14	6·044	−4	+14	5·693	−4	+14

Values of $R(X, T)$

T	X 1·0	1·2	1·4	1·6	1·8	2·0	2·2	2·4	2·6	2·8	3·0
5	1·986	1·727	1·492	1·280	1·090	0·921	0·7705	0·6388	0·5240	0·4249	0·3401
6	2·294	2·025	1·779	1·553	1·349	1·164	0·9986	0·8505	0·7190	·6031	·5016
7	2·581	2·304	2·048	1·812	1·596	1·399	1·2203	1·0585	0·9129	·7827	·6669
8	2·852	2·567	2·303	2·059	1·833	1·625	1·4353	1·2619	1·1043	0·9618	·8334
9	3·107	2·818	2·547	2·295	2·060	1·844	1·6440	1·4605	1·2924	1·1391	0·9998
10	3·351	3·056	2·779	2·521	2·279	2·055	1·8467	1·6543	1·4770	1·3142	1·1652
11	3·584	3·285	3·003	2·738	2·491	2·259	2·0436	1·8434	1·6579	1·4866	1·3289
12	3·808	3·504	3·218	2·948	2·695	2·457	2·2353	2·0280	1·8352	1·6562	1·4906
13	4·023	3·716	3·426	3·152	2·893	2·650	2·4219	2·2083	2·0088	1·8230	1·6502
14	4·231	3·921	3·627	3·349	3·086	2·837	2·6039	2·3846	2·1791	1·9869	1·8075
15	4·432	4·119	3·822	3·540	3·273	3·020	2·7816	2·5570	2·3459	2·1479	1·9626
16	4·627	4·312	4·012	3·726	3·455	3·198	2·9551	2·7257	2·5096	2·3063	2·1154
17	4·817	4·499	4·196	3·907	3·633	3·372	3·1248	2·8910	2·6702	2·4619	2·2659
18	5·001	4·681	4·375	4·084	3·806	3·542	3·2910	3·0530	2·8278	2·6150	2·4143
19	5·180	4·859	4·551	4·256	3·976	3·708	3·4537	3·2120	2·9827	2·7656	2·5604
20	5·356	5·032	4·722	4·425	4·142	3·871	3·6132	3·3679	3·1349	2·9138	2·7044

T	X 3·2	3·4	3·6	3·8	4·0	4·2	4·4	4·6	4·8	5·0
5	0·2682	0·2081	0·1583	0·1178	0·0852	0·0596	0·0398	0·0250	0·0142	0·0067
6	·4133	·3371	·2720	·2168	·1704	·1319	·1004	·0748	·0545	·0385
7	·5643	·4742	·3954	·3270	·2680	·2176	·1748	·1388	·1089	·0842
8	·7184	·6159	·5249	·4446	·3741	·3126	·2593	·2134	·1742	·1408
9	0·8738	·7602	·6582	·5672	·4862	·4145	·3513	·2961	·2479	·2062
10	1·0292	0·9057	0·7939	0·6930	0·6023	0·5212	0·4489	0·3848	0·3282	0·2784
11	1·1841	1·0516	0·9307	0·8208	·7213	·6315	·5507	·4783	·4136	·3562
12	1·3377	1·1971	1·0680	0·9499	·8422	·7443	·6555	·5753	·5031	·4383
13	1·4900	1·3419	1·2052	1·0796	0·9643	·8588	·7626	·6751	·5958	·5241
14	1·6406	1·4856	1·3421	1·2094	1·0870	0·9745	·8714	·7770	·6909	·6126
15	1·7895	1·6282	1·4782	1·3389	1·2100	1·0910	0·9813	0·8805	0·7880	0·7035
16	1·9366	1·7694	1·6134	1·4681	1·3331	1·2079	1·0920	0·9851	0·8866	·7961
17	2·0818	1·9091	1·7475	1·5966	1·4558	1·3249	1·2033	1·0906	0·9864	·8902
18	2·2252	2·0474	1·8806	1·7243	1·5782	1·4418	1·3147	1·1966	1·0870	0·9855
19	2·3667	2·1841	2·0124	1·8511	1·6999	1·5584	1·4262	1·3029	1·1882	1·0815
20	2·5064	2·3193	2·1430	1·9771	1·8211	1·6747	1·5376	1·4095	1·2898	1·1783

Table of $Q(X, T)$

T	Q	δ_T^2	δ_X^2	Q	δ_T^2	δ_X^2	Q	δ_T^2	δ_X^2	Q	δ_T^2	δ_X^2	Q	δ_T^2	δ_X^2
X	0·0			0·2			0·4			0·6			0·8		
5	25·55	70	61	21·45	69	55	17·89	69	49	14·83	68	44	12·22	66	39
6	33·20	64	66	28·33	64	60	24·05	63	54	20·32	62	49	17·07	61	44
7	41·50	59	71	35·84	59	65	30·84	59	59	26·42	58	54	22·54	57	48
8	50·38	56	76	43·95	55	69	38·21	55	63	33·11	54	58	28·58	54	53
9	59·82	52	80	52·61	52	74	46·14	52	68	40·34	52	62	35·16	51	57
10	69·79	50	84	61·80	50	78	54·58	49	72	48·08	49	66	42·24	48	61
11	80·26	48	88	71·48	48	81	63·52	47	75	56·32	47	70	49·81	46	64
12	91·20	46	91	81·64	46	85	72·94	45	79	65·02	45	73	57·84	44	68
13	102·59	44	95	92·26	44	88	82·80	44	82	74·18	43	77	66·31	43	71
14	114·43	42	98	103·31	42	92	93·11	42	86	83·76	42	80	75·22	41	75
15	126·68	41	101	114·78	41	95	103·83	41	89	93·77	40	83	84·54	40	78
16	139·35	40	105	126·66	40	98	114·96	39	92	104·17	39	86	94·26	39	81
17	152·41	38	108	138·94	38	101	126·48	38	95	114·97	38	89	104·36	38	84
18	165·86	37	111	151·60	37	104	138·39	37	98	126·16	37	92	114·85	37	87
19	179·68	36	114	164·64	36	107	150·67	36	101	137·71	36	95	125·70	36	90
20	193·86	35	116	178·03	35	110	163·31	35	104	149·62	35	98	136·91	35	93

Values of $Q(X, T)$

T	X 1·0	1·2	1·4	1·6	1·8	2·0	2·2	2·4	2·6	2·8	3·0
5	9·99	8·10	6·52	5·19	4·10	3·20	2·47	1·88	1·414	1·045	0·758
6	14·27	11·86	9·79	8·03	6·54	5·29	4·24	3·37	2·658	2·073	1·599
7	19·15	16·19	13·62	11·40	9·49	7·85	6·46	5·28	4·290	3·458	2·767
8	24·58	21·06	17·97	15·27	12·92	10·88	9·12	7·60	6·308	5·203	4·267
9	30·54	26·45	22·82	19·63	16·82	14·35	12·20	10·33	8·705	7·304	6·100
10	37·00	32·32	28·15	24·44	21·16	18·25	15·69	13·44	11·475	9·758	8·265
11	43·94	38·67	33·94	29·70	25·93	22·57	19·58	16·94	14·610	12·559	10·760
12	51·33	45·46	40·16	35·39	31·11	27·28	23·86	20·81	18·104	15·702	13·579
13	59·17	52·68	46·80	41·49	36·70	32·39	28·52	25·05	21·949	19·182	16·721
14	67·42	60·32	53·86	48·00	42·68	37·88	33·55	29·64	26·137	22·992	20·179
15	76·09	68·36	61·31	54·88	49·04	43·74	38·93	34·59	30·663	27·128	23·949
16	85·15	76·79	69·14	62·15	55·77	49·96	44·67	39·87	35·519	31·582	28·028
17	94·59	85·60	77·35	69·79	62·86	56·53	50·75	45·49	40·699	36·351	32·409
18	104·41	94·78	85·92	77·78	70·30	63·44	57·17	51·43	46·197	41·428	37·090
19	114·59	104·32	94·85	86·12	78·08	70·69	63·91	57·70	52·008	46·809	42·065
20	125·13	114·22	104·12	94·80	86·20	78·27	70·98	64·28	58·126	52·489	47·330

T	X 3·2	3·4	3·6	3·8	4·0	4·2	4·4	4·6	4·8	5·0
5	0·538	0·372	0·250	0·162	0·100	0·058	0·030	0·014	0·004	0·000
6	1·218	0·916	0·678	0·494	0·353	0·247	·168	·111	·070	·043
7	2·195	1·726	1·344	1·036	0·790	0·594	·441	·323	·232	·163
8	3·477	2·816	2·264	1·807	1·431	1·123	0·874	0·673	·513	·387
9	5·070	4·191	3·446	2·818	2·290	1·849	1·484	1·182	0·934	0·732
10	6·973	5·857	4·898	4·078	3·378	2·784	2·283	1·862	1·509	1·216
11	9·186	7·814	6·623	5·591	4·701	3·937	3·282	2·724	2·250	1·849
12	11·708	10·063	8·621	7·362	6·265	5·312	4·488	3·777	3·166	2·643
13	14·536	12·602	10·895	9·391	8·071	6·915	5·905	5·027	4·265	3·605
14	17·667	15·430	13·442	11·680	10·122	8·748	7·539	6·479	5·551	4·741
15	21·097	18·544	16·262	14·228	12·419	10·813	9·392	8·136	7·030	6·057
16	24·824	21·942	19·354	17·036	14·962	13·112	11·465	10·001	8·704	7·557
17	28·843	25·621	22·715	20·100	17·751	15·645	13·760	12·077	10·577	9·243
18	33·150	29·577	26·343	23·421	20·785	18·412	16·278	14·364	12·650	11·118
19	37·742	33·809	30·237	26·997	24·063	21·412	19·019	16·864	14·925	13·185
20	42·615	38·313	34·392	30·825	27·585	24·645	21·983	19·576	17·403	15·445

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